The Utility Value of Longevity Risk Pooling: Analytic Insights

Moshe A. Milevsky[1] and Huaxiong Huang

10 March 2018 (With Technical Appendix)

[1]Milevsky (the contact author) is professor of finance at the Schulich School of Business and can be reached via email at: milevsky@yorku.ca, or at Tel: 416-736-2100 x 66014. His address is: 4700 Keele Street, Toronto, Ontario, Canada, M3J 1P3. Huaxiong Huang is a Professor of Mathematics at York University and the Deputy Director of the Fields Institute. The authors thank the editor of NAAJ and two anonymous reviewers for helpful comments on an earlier draft. Milevsky acknowledges a series of (mostly) interesting conversations with Rowland Davis, Evan Ingles and Steven Siegel at the U.S. *Society of Actuaries*, which motivated the writing of this paper, and would especially like to thank Daniel Bauer for encouraging comments.

Abstract

The Utility Value of Longevity Risk Pooling: Analytic Insights

The consensus among scholars is that (some) longevity risk pooling is the optimal strategy for drawing down wealth in retirement and a robust literature has developed around its measurement via *annuity equivalent wealth*. However, most of the published work is conducted numerically and authors usually report only a handful of limited values. In this paper we derive some closed-form expressions for the value of longevity risk pooling with fixed life annuities under constant relative risk aversion preferences. We show, for example, that this value converges to $\sqrt{e} - 1 \approx 65\%$, when the interest rate is the inverse of life expectancy, lifetimes are exponentially distributed and utility is logarithmic. In general the various formulae we derive match previously published numerical results, when properly calibrated to discrete time and tables. More importantly, we focus attention on the incremental utility from annuitization when the retiree is already endowed with pre-existing pension income such as Social Security benefits. Indeed, due to the difficulty in working with the so-called *wealth depletion time* in lifecycle models, we believe this is an area that hasn't received proper attention from actuarial researchers. Our paper offers tools to explain the value of longevity risk pooling.

1 Motivation and Outline

Researchers and scholars have long known – since the original work by Yaari (1965) – that longevity risk pooling in the form of life annuities (a.k.a. actuarial notes, or instantaneous tontines) is the optimal strategy for drawing down wealth in retirement. In an idealized (utopian) world with no market imperfections, fair insurance pricing and zero bequest motives, the optimal allocation to annuities (anytime, anywhere) is 100%. Even when loadings, imperfections and bequest motives are added to the basic model, the work by Davidoff et al. (2005) argues for a substantial allocation to life annuities.

A large and very robust literature has developed around the measurement of so-called annuity equivalent wealth (AEW), a metric used in a variety of widely cited-papers starting with Kotlikoff and Spivak (1981) and continued by Brown (2001). However, most of the AEW-based research is conducted numerically and authors usually report only a handful of limited values in their papers – for understandable reasons, we might add. Nevertheless a diligent reader is hard pressed to solve or replicate their own stochastic dynamic program if they seek personalized numbers. This is becoming increasingly important in a scholarly environment which places (greater) value on replicability and transparency. In addition to the academic need, industry practitioners have also struggled to explain the *value* of longevity risk pooling to a wider public.

Motivated by the need for computational simplicity in this paper we derive a variety of analytic expressions for the AEW under constant relative risk aversion (CRRA) consumption preferences. We provide a complete characterization under exponential mortality as well as results for more general models of mortality, such as Gompertz-Makeham (GM). And, since any mortality table relevant to retirees can be calibrated to the GM law with little economic error, our expressions can also be used to estimate the value of pooling in any environment, which should be of practical use to actuaries.

We note up front that the output from our analytic expressions roughly match previously published numbers which were based on discrete time and tables, so there are no surprises, conflicts or inconsistencies with the prior annuity economics literature. Rather, our pedagogical simplification enables users to calculate and report the utility *value of longevity risk pooling* under different mortality assumptions, retirement ages, discount rates and levels of risk aversion.

From a risk theory or conceptual point of view, our work highlights the often small incremental utility that comes from annuitization when the individual is already endowed with (substantial) pre-existing pension income in the form of Social Security benefits or income from a Defined Benefit (DB) pension plan. Methodologically, we compute the annuity equivalent wealth without having to resort to discounting the present value of pension benefits, adding them to wealth or ignoring them altogether; which is often the case in the economics literature.

The benefit of <u>additional</u> longevity risk pooling – and the proper computation of the AEW with background pension income – is an item that hasn't received as much attention from actuarial researchers and annuity scholars. For example, the book by Sheshinski (2007) makes no reference to this matter, mainly because he is concerned with the optimal strategy and annuity valuation in the absence of pre-existing pensions. The more financially oriented lifecycle and portfolio choice literature, such as Cocco and Gomes (2012) for example, aren't concerned with computing AEW. They focus on explaining observed behavior, so these issues aren't addressed explicitly either.

1.1 Plan and Agenda

The remainder of this paper is organized as follows. In the next section (#2) we review the conceptual framework, formally define the annuity equivalent wealth and explain its relation to the value of longevity risk pooling, a term which is more familiar to actuaries and insurance specialists. In section (#3) we restrict our attention to the case in which the remaining lifetime random variable is exponentially distributed, which means that the mortality hazard rate is constant. Although this is a (very) unrealistic assumption (especially to actuaries), it is used in many insurance economic papers. In the same section (#3) we offer a range of numerical results or case studies in the presence of pre-existing pension income. Section (#4) moves on to the more general mortality models. We obtain a closed-form expression for the value of pooling in the absence of pre-existing pension income as a function of the ratio of two annuity factors. If the annuity factors themselves can be expressed analytically, then so can the value of longevity risk pooling. Our expressions and results are compared with the (numerical) values presented in Brown et al. (2001), and are in fact consistent with their results in the absence of pre-existing pension income.

One of the main theoretical contributions in this paper is to argue that when pre-existing (fixed) pensions are included in a lifecycle model one has to be (very) careful about how to define annuity equivalent wealth and the value of longevity risk pooling. This is due to the *wealth depletion time* which complicates the discounted utility analysis. Although the concept of a *wealth depletion time* is explained in Leung (2002, 2007) or in Lachance (2012), within the context of the Yaari (1965) lifecycle model, it doesn't appear to be well known[1]. We will explain the implication of this to the utility valuation of AEW. Finally, section (#5) concludes the paper with a summary of the main results and expressions. Note that stand-alone code (R script) with an algorithm that can be used to generate numerical results under a variety of assumptions are in a technical appendix to the article.

2 Annuity Equivalent Wealth: Derived

2.1 Notation and Terminology

Let the pair (w, π) denote an initial retirement endowment of liquid (non annuitized) wealth w, plus pension income denoted by π, which measures an annual cash-flow in real terms

[1] For example, Leung (2002) writes that: *"Previous studies utilizing Yaari's (1965) model of uncertain lifetime...have failed to recognize that terminal wealth depletion is an intrinsic and important property of the model...erroneous and misleading results will be obtained if it is ignored in the investigation"*, pg. 582

beginning immediately ($t = 0$) and continuing until death. The non-pensionized wealth w, is assumed to be invested and growing at a real risk-free rate denoted by r. That account will be the source of all consumption above and beyond what is flowing from the pension income π. The economic value of the total endowment is: $w + PV(\pi)$, where $PV(\pi)$ denotes the actuarial present value of pre-existing pension annuity income using the appropriate mortality basis. We occasionally use the variable $\psi = PV(\pi)/(w + PV(\pi))$, which denotes the fraction of the personal balance sheet that is *pensionized.* And, while it is clear to pension economists that *ceteris paribus* it is better to have a value of ψ as close to one as possible, the question of how to optimize consumption given an initial endowment (w, π) affects the utility value of additional annuitization.

Later we formally define *maximized* discounted lifetime utility for the given pair (w, π) but at this juncture we introduce and denote it by $U^*(w, \pi)$. We add the x subscript to $U^*_x(w, \pi)$ if and when we need to draw attention to the individual's current age x. Remember that in the background of $U^*(w, \pi)$ resides an optimal consumption strategy denoted by c^*_t, beginning at time $t = 0$ until death, which dictates how w, is spent over time. There are as many possible values of utility $U(w, \pi)$ as there are strategies c_t, but there is only one unique $U^*(w, \pi)$ and corresponding strategy c^*_t.

Our notation for a liquid wealth plus pension income endowment pair (w, π) is used by Cannon and Tonks (2008) for example, and implicitly in Brown (2001) as well as other papers that measure utility value of annuities. Just to clarify notation though, $U_{65}(100, 3)$ without the asterisk could represent one possible utility value at the age of $x = 65$, of someone who starts retirement with an initial wealth of $w = 100$ dollars plus a pension entitlement of $\pi = 3$ dollars who withdraws 4% of her initial investable wealth, that is $c_t = 4+3 = 7, t \geq 0$, dollars each year until death (or the money runs out). It's a feasible consumption strategy but not the optimal one.

We now move on to voluntary annuitization. The individual can convert (some or all) of his/her initial wealth w into additional pension income by purchasing annuities at a unit price a_x. The actuarial present value of the pre-existing pension income is $PV(\pi) = a_x\pi$ and $\psi = a_x\pi/(w + a_x\pi)$. Likewise, spending a_x units at age x, will entitle the retiree to one additional \$1 of income. We make an assumption that pre-existing pension income π and voluntary annuity purchases are priced and valued the same. They are substitutes, notwithstanding frictions encountered in replacing w with more pension income.[2]

If one think's of a_x as the frictionless pension annuity factor and $(1 + \kappa)a_x$ as the loaded annuity factor relevant to additional annuity purchases, then most of what we do in this paper assumes $\kappa = 0$. Practically speaking, for every \$1 of liquid investable wealth (w) that is annuitized, leaving $(w - 1)$ to invest at r, pension income will increase by $1/(1 + \kappa)a_x$. Of course an individual endowed with (w, π) can convert his/her entire holdings into $(0, \pi + w/(1 + \kappa)a_x)$ if they so choose, but we do not allow transfer in the other direction; that is to unwind or sell back pension annuities. In other words, they can't convert the pair (w, π) into $(w + \pi(1 + \kappa)a_x, 0)$. But nor would they want to, as Menachem Yaari argued over 50 years ago in Yaari (1965), assuming $\kappa = 0$. Again, for most of what follows, $\kappa = 0$.

[2]See Bodie (1990) or Blake (1999) for more on pensions as "cheap" longevity insurance.

2.2 Definition and Distinction

We now get to the crux of the matter and define the so-called *annuity equivalent wealth* (AEW). Even though we haven't yet written down a formal expression for discounted lifetime utility – the value function in the language of dynamic programming – or proven the optimality of pooling longevity risk, our starting point is that:

$$U_x^*(w,\pi) \;\leq\; U_x^*(w-\epsilon,\pi+\epsilon/a_x), \tag{1}$$

which can be proven under general utility preferences, but only in frictionless markets. Indeed, the recent work by Reichling and Smetters (2015) questions the frictionless assumption and hence the validity of equation (1). Intuitively though, in the absence of frictions, (maximized) utility is greater with more (vs. less) annuities. Every ϵ that is spent on annuities will increase maximal utility. This again is the essence of the Yaari (1965) argument for converting all wealth into annuities, since $U^*(0,\pi+w/a_x)$ will provide the greatest amount of utility, assuming no loading, no fees and no anti-selection. We will return to these assumptions ($\kappa=0$) in the concluding remarks. Note that in the background of equation (1), every initial combination of (w,π) and its corresponding maximized utility $U^*(w,\pi)$ will involve a different consumption strategy c_t^*, which is central to what follows later.

The formal definition of annuity equivalent wealth is the quantity $\hat{w}$ that satisfies the following equation:

$$U_x^*(\hat{w},\pi) \;=\; U_x^*(0,\pi+w/a_x), \tag{2}$$

where w is the initial liquid wealth of the retiree. The markup of $\hat{w}$ over w, reflects the amount someone (with wealth w) would require in compensation to have the same utility level as someone who annuitizes w. For example, Brown (2001) argues that consumers with larger $\hat{w}/w$ values, were actually more likely to purchase additional annuities at retirement. In what follows we will focus on $\hat{w}/w-1$, and refer to this as the value of longevity risk pooling *in-the-large*, because it assumes the entire w is annuitized.

This paper is concerned with describing a variety of techniques and algorithms for isolating the value of δ that satisfies the equation:

$$U_x^*((1+\delta)w,\pi) \;=\; U_x^*(0,\pi+w/a_x), \tag{3}$$

We use the phrase value of pooling *in the large*, because it assumes the consumer convert all of their wealth w into more annuity income. We will later contrast this quantity δ with a parallel metric *in the small* which measures the incremental value of converting only \$1 into additional annuity income. And, while the CRRA assumption might tempt readers into believing that the value of pooling for one incremental dollar is the same as the value of pooling for the entire w, the fact is the homogeneity breaks down due to the wealth depletion time. Also, we operate in the classical utility framework and will not veer into a discussion of behavioral economics as it relates to the *framing* of annuities – investment vs. consumption – or the reasons for the so-called annuity puzzle. Rather, we carry the torch of Daniel Bernoulli and refer readers to Brown et al. (2008) for that aspect of annuity demand.

The value of pooling $\delta = \hat{w}/w - 1$ will depend on a large number of explicit and implicit variables that are buried in the utility calculations, such as risk aversion and subjective discount rates, the specific mortality table being used (subjectively or objectively) and the interest rate used for pricing the annuity factor a_x. If we wanted to clutter-up the δ, we would add all those parameters as explicit arguments. But, that dependence is well known to researchers. What is less known (or at least emphasized) is that the value of pooling δ also depends on the amount of pre-existing pension income π in the initial endowment pair.

Of course, authors such as Brown (2001) make it very clear that the fraction of total wealth that is pre-annuitized (we used the symbol ψ to denote this) will impact the AEW and that it's lower for individuals with pre-existing pension income. Nevertheless, it is often obscured within the dynamic programing calculations and rarely treated as a separate component or argument in the AEW value. The exact mechanism by which π reduces the AEW is, for the most part, glossed over.

For example, it is incorrect to capitalize or compute the actuarial present value of the annuity income π, add this to initial wealth w and then use the aggregate (new) number $w + PV(\pi)$ to compute AEW values. It will overstate the AEW. Methodologically, it is equally erroneous to ignore the pre-existing pension income π and compute the AEW for the w alone, under the misguided assumption that utility is CRRA and hence everything scales. On a more subtle level, adding preexisting annuity income into the felicity function's consumption argument – and effectively moving from a CRRA formulation into a HARA world – also ignores the issue by forcing a non-zero discretionary consumption at the boundary. Stated formally:

$$
\begin{aligned}
U^*_x((1+\delta)w, \pi) &\neq U^*_x((1+\delta)w + \pi a_x, 0) \\
U^*_x((1+\delta)w, \pi) &\neq U^*_x((w + \pi a_x)(1+\delta), 0) \qquad (4)
\end{aligned}
$$

The pre-existing pension income π alters the optimal consumption path c^*_t in a very critical way that feeds back into utility. And, since this is a large part of our story, we will occasionally use the subscript δ_ψ to remind readers that the AEW (also) depends on the fraction of wealth that is pre-annuitized.

Ceteris paribus, we know that: $\delta_{(\psi+\epsilon)} - \delta_\psi > 0$. In the frictionless Yaari (1965) setting it is always worth paying for a little bit more of annuity income. The issue and impetus for this paper is to arrive at an expression for δ_ψ so that this marginal or incremental benefit can be properly measured quickly, easily and under a wide variety of parameters. By its very definition in equation (3), the value δ_ψ assumes the entire w wealth is used to purchase (more) annuities. But what is the *incremental* value of longevity pooling?

2.3 Pooling in the small

In the presence of pre-existing pension income π – whether it be Social Security benefits or voluntarily purchased annuity income – we are forced to *refine* the meaning of *annuity equivalent wealth* and the corresponding value of longevity risk pooling.

Indeed, when the consumer or retiree is (only) endowed with liquid investable assets ($w > 0, \pi = 0$), and assuming they exhibit constant relative risk aversion utility preferences, the δ_0 value scales in wealth. The value of $\hat{w}$ for someone with $w = \$1$ (divided by w) is identical to the AEW of a wealthier retiree with an initially endowed $w = \$1$ million (divided by w.) In fact, this is precisely why virtually all of the authors in the annuity economics literature express and report AEW numbers relative to a dollar value. Authors arbitrarily assume an initial wealth of $w = \$1$, because it makes no difference to δ. The classical value of longevity risk pooling doesn't depend on the actual value of w, assuming consumers (i.) obey CRRA utility preferences, (ii.) have no exogenous pension income, (iii.) have no bequest motives, and (iv.) are in frictionless markets.

However, once an exogenous or pre-existing pension income is *properly* introduced into the life-cycle consumer optimization problem, the value of w as it relates to π, actually matters. This is not usually the case with the convenient homogenous specification of utility and might be one of the reasons it has been overlooked in the literature. Ignoring this fact will *overstate* the AEW and the value of pooling because $\delta_0 > \delta_\psi$, for any value of $\psi > 0$, that is when the individual already has pre-existing pension annuity income. The value of $\delta_{0.99}$, which represents someone for whom the majority of wealth is already pensionized, should intuitively be quite small.

We therefore define an additional metric for the value of longevity risk pooling, denoted by v which captures the value for one *incremental* dollar of wealth. In some sense, it's back to a marginal analysis. Technically, the AEW *in-the-small* addresses the following: Assume a retiree has $w = \$100,000$ in investable retirement assets and $\pi = \$10,000$ in pre-existing pension income which induces lifetime utility denoted by: $U_x^*(w, \pi)$.

Now, they are about to spend only \$1 from their $w = \$100,000$ to purchase incremental pension income beyond the $\pi = \$10,000$ they already are entitled to. This will obviously leave them with only $\$99,999$ in investable retirement wealth plus revised pension income of $\$10,000 + 1/a_x$, where a_x is the standard annuity factor at age x. How much additional wealth would the retiree – who didn't spend the additional \$1 to purchase additional pension annuity income – require to induce the same level of utility? Needless to say, the answer will depend on the coefficient of relative risk aversion, but will actually change if the initial wealth were $w = \$1,000,000$ or only $\$50,0000$, or the pension π was different.

For clarity we will report and display both δ values (in-the-large) and v values (in-the-small). We solve for v by equating levels of optimal utility, via the following equation:

$$U_x^*(w + v, \pi) \ = \ U_x^*(w - 1, \pi + 1/a_x). \tag{5}$$

We now move on to the computations.

2.4 Computing Utility

We are at the point in the narrative where we can present a formal expression for discounted lifetime utility: $U_x^*(w, \pi)$. Let $u(c)$ denote a constant relative risk aversion (CRRA) utility (a.k.a. felicity) function parameterized by risk aversion γ, and a subjective discount rate ρ. Formally, $u(c) = c^{1-\gamma}/(1-\gamma)$. The maximal utility is:

$$U_x^*(w,\pi) \;= \max_{c_t} \int_0^\infty e^{-\rho t}\,({}_tp_x)\,u(c_t)\,dt, \tag{6}$$

with a dynamic budget constraint determined by:

$$dW_t \;=\; (rW_t \,+\, \pi -\, c_t)\,dt, \;\; W_0 = w. \tag{7}$$

The objective function (6) and constraint (7) is part of the classical lifecycle framework which will soon celebrate its centenary anniversary, although recently Bommier (2006) questioned the underlying risk neutrality assumption and proposes a different form altogether. We follow Yaari (1965), Levhari and Mirman (1977), Davies (1981), Butler (2001) or Pashchenko (2013) where this framework is used to extract testable implications for rational behavior with lifetime uncertainty and annuities. The pension income π flows into the (one) account earning r, and then consumption c_t is extracted from the same account. By definition the optimal consumption function is c_t^* and the difference: $(c_t^* - \pi)$ is the net spending rate from liquid wealth, whereas $(c_t^* - \pi)/W_t$ is the spending rate as a fraction of current wealth. Indeed, saving might continue into retirement and wealth might continue to grow temporarily, for someone with a sufficiently low discount rate ρ. We will not allow any borrowing (against future pension income) so that wealth $W_t \geq 0$ at all times.

Also, to be consistent with the literature we assume for most of what follows that $\rho = r$ and the subjective discount rate is equal to the interest rate earned in the account. The only reason to prefer early vs. late consumption, in our model, is due to mortality beliefs and the inter-temporal elasticity of substitution, $1/\gamma$ in our model.

Moving on, without any loss of generality we can decompose or break-up the integral in equation (6) into two arbitrary parts as follows:

$$U_x^*(w,\pi) \;= \max_{c_t} \left[\int_0^\tau e^{-rt}\,({}_tp_x)\,u(c_t)dt \;+\; \int_\tau^\infty e^{-rt}\,({}_tp_x)\,u(c_t)dt\right], \tag{8}$$

where the break takes place at time τ. It might seem odd to split up the objective function in this manner, but in fact when $\pi > 0$, there is a qualitative change in optimal consumption at some point during the horizon $t \in [0,\infty)$. That is, liquid wealth is actually depleted and the optimal consumption rate $c_t^* = \pi$ from that point onward. That is the τ value we select. Until that *wealth depletion time* consumption is sourced from from both pension income and wealth. But after $t \geq \tau$ consumption is exactly equal to the pension, the individual has run out of liquid (non-annuitized) funds and $W_t = 0$, for $t \geq \tau$.

To be crystal clear we are not imposing this on the problem. It actually *is* the optimal policy, as elaborated on by Leung (2002, 2007) and carefully explained in the (textbook) by Charupat, et al. (2012), chapter #13. We can't emphasize enough how critical this (seemingly minor) point is to the calibration of lifecycle models in general and the computation of AEW values in particular. At the risk of flogging a dead horse, if one assumes all pension income is capitalized and discounted to time zero, or if the pension income is added to optimal consumption as a scaling afterthought, the *wealth depletion time* will be lost in the backward induction algorithm.

Technically, the value of $c_t^* = \pi$ for $t \geq \tau$ and therefore $u(c_t^*)$ from the point of $t = \tau$ onward is constant. This enables us to take the next step and express the objective function (yet again) and optimal utility as:

$$U_x^*(w, \pi) \ = \max_{\tau,\, c_t} \left[\int_0^{\tau} e^{-rt} \, ({}_tp_x) \, u(c_t) dt \right] \ + \ u(\pi) \, a_x(\tau), \tag{9}$$

where $a_x(\tau)$ is the (deferred) annuity factor at age x, but beginning or starting income at time τ. It is sometimes called an advanced life delayed annuity (ALDA). As far as notation is concerned, when the deferral period is $\tau = 0$, we will resort to the simpler expression a_x instead of the cumbersome $a_x(0)$. More importantly, when $w = 0$ this all collapses to:

$$U_x^*(0, \pi) \ = \ u(\pi) \, a_x, \tag{10}$$

If the consumer has no investable funds ($w = 0$) and they are living-off pension π income only, then discounted (optimal) lifetime utility is simply: $u(\pi) \, a_x$. See Cannon and Tonks (2008) for a derivation of equation (10), even under more general (Epstein Zin) preferences. When all wealth $w > 0$ is converted to annuities, utility is: $u(w/a_x + \pi)a_x$.

Finally, the optimal consumption before the wealth depletion time τ, when consumption is not equal to the pension π, is:

$$c_t^* \ = \ c_0^*({}_tp_x)^{1/\gamma} \ = \ \left(\frac{\pi}{({}_\tau p_x)^{1/\gamma}} \right) ({}_tp_x)^{1/\gamma}, \ \text{ when } \pi > 0 \tag{11}$$

where the optimal initial consumption rate c_0^* is related to (terminal) consumption π, via the relationship $c_0^* = \pi/({}_\tau p_x)^{1/\gamma}$, as long as there actually is some pension income $\pi > 0$. In contrast to equation (11), when $\pi = 0$, the relevant consumption rate must be sufficient to last forever (in theory), so that $w = c_0^* \int_0^\infty e^{-rt}({}_tp_x)^{1/\gamma} dt$, which leads to the corresponding:

$$c_t^* \ = \ \left(\frac{w}{\int_0^\infty e^{-rt}({}_tp_x)^{1/\gamma} dt} \right) ({}_tp_x)^{1/\gamma}, \ \text{ when } \pi = 0. \tag{12}$$

We simply note that the integral in the denominator of equation (12) is an annuity factor (of sorts) assuming that the survival probability $({}_tp_x)$ is shifted or distorted by $1/\gamma$. For example, when $\gamma = 1$ and utility is logarithmic, the optimal consumption function c_t^* in equation (12) collapses to the hypothetical annuity consumption w/a_x times the survival probability $({}_tp_x)$, which is clearly less than what a true annuity would have provided. The individual who converts all liquid wealth w into the annuity would consume $c_t^* = w/a_x$ for ever, but the non-annuitizer must continue to reduce consumption in proportion to their survival probability as a precautionary measure. Although it might seem as if equation (11) or equation (12) is plucked-out of thin air, neither of these are new or novel. See Cannon and Tonks (2008) or Charupat, et al. (2012), chapter #13 for example. Rather, our objective here is to use these analytic expressions to solve for and extract δ, which has not been done previously in the literature.

2.5 Solving for δ when $\pi = 0$.

Using the prior notation, let $U_x^*(w,0)$ denote discounted lifetime utility of wealth w without annuities. This can be expressed mathematically as:

$$U_x^*(w,0) \;=\; \int_0^\infty e^{-rt}\,({}_tp_x)\,u(c_t^*)\,dt, \tag{13}$$

where c_t^* is the optimized consumption path we displayed in equation (12). Note that there is no wealth depletion time since there are no annuities. In contrast let $U_x^*(0,w/a_x)$ denote discounted lifetime utility of wealth, assuming wealth w **is entirely** annuitized or pooled at age x. Discounted utility is:

$$U_x^*(0,w/a_x) \;=\; \int_0^\infty e^{-rt}\,({}_tp_x)\,u(w/a_x)\,dt, \tag{14}$$

where the optimized consumption path is trivially $c_t^* = w/a_x$, for all t. Technically $\tau = 0$ because all wealth has been annuitized at time zero.

We won't belabor the point that $U_x^*(0,w/a_x) \geq U_x^*(w,0)$ but simply conclude by noting that δ_0 (assuming pension income $\pi = 0$ so that the $\psi = 0$) will satisfy the following equation:

$$U_x^*((1+\delta_0)w,0) \;= U_x^*(0,w/a_x), \tag{15}$$

and the rest is algebra. Basically, we compute the inverse function $y = U_x^{(*,-1)}(z,\pi)$ for $z = U_x^*(y,\pi)$ with respect to the wealth variable w. The quantity we are looking for is:

$$\delta_0 \;=\; \frac{1}{w}U^{(*,-1)}(U_x^*(0,w/a_x),0) \;-1, \tag{16}$$

and more generally, in the presence of pension income $\pi > 0$, so that the fraction of balance sheet pre-annuitized is: $\psi = a_x\pi/(w+a_x\pi) > 0$, the expression generalizes to:

$$\delta_\psi \;=\; \frac{1}{w}U^{(*,-1)}(U_x^*(0,w/a_x+\pi),0) \;-1, \tag{17}$$

where the (inverse) value function in the denominator of equation (17) includes and requires the calculation of the wealth depletion time τ. Both of these, δ_0 and δ_ψ for $\psi > 0$ are concerned with complete annuitization, or what we called AEW *in-the-large*. If we focus on the incremental dollar of annuitization, the relevant expression becomes:

$$\upsilon \;=\; U^{(*,-1)}(U_x^*(w-1,1/a_x+\pi),\pi) \;-w, \tag{18}$$

where υ is (what we call) the value *in-the-small*. The remainder of this paper makes specific assumptions on the underlying mortality function $({}_tp_x)$, pre-existing pension income π, and then inverts the value functions to obtain expressions for equations (16) - (18).

3 Exponential Remaining Lifetime

3.1 Start with the Corners

If the entire w is annuitized at time zero, the optimal consumption rate is: $c_t^* = (w/a_x + \pi)$, which in the case of an exponentially distributed lifetime collapses to: $c_t^* = w(r+\lambda) + \pi$, because the annuity factor is: $a_x = 1/(r+\lambda)$. The maximal utility possible when the entire endowment (w, π) is annuitized, according to equation (14), is:

$$U_\lambda^*(0, w/a_x) \;=\; \int_0^\infty e^{-rt}(e^{-\lambda t})\frac{(w(r+\lambda))^{1-\gamma}}{1-\gamma}dt \;=\; \frac{(w(r+\lambda))^{1-\gamma}}{(1-\gamma)(r+\lambda)}, \tag{19}$$

where the subscript λ in the U_λ reminds readers we are operating under an exponential remaining lifetime in which the (only) parameter that matters is the hazard rate λ. The first item in the integrand captures the subjective discounting of utility, the second is the survival probability and the third is the instantaneous utility, which is constant (because consumption equals the annuity).

Equation (19) represents the gold standard by which all other strategies will be measured. It represents the highest possible level of utility, assuming the retiree converts the entire w to additional annuity units. At the other extreme of equation (19) is the (obstinate) individual who refuses to annuitize any wealth at all, and finances all consumption from liquid wealth. For this individual the maximal achievable level of utility can be simplified to:

$$U_\lambda^*(w, 0) \;=\; \int_0^\infty e^{-(r+\lambda)t}\,\frac{(c_t^*)^{1-\gamma}}{1-\gamma}dt \;=\; \frac{(w(r+\lambda/\gamma))^{1-\gamma}}{(1-\gamma)(r+\lambda/\gamma)}. \tag{20}$$

Note from equation (12) that $({}_tp_x)^{1/\gamma} = e^{-(\lambda/\gamma)t}$ and the optimal consumption function $c_t^* = w(r+\lambda/\gamma)e^{-(\lambda/\gamma)t}$. By comparing equation (19) to equation (20) and the concavity of the power function, one can see explicitly that as long as the mortality rate $\lambda > 0$, the utility from annuitization $U_\lambda^*(0, w/a_x)$ is greater than the utility from self-annuitization $U_\lambda^*(w, 0)$, or any other systematic withdrawal or drawdown plan.

Equating the utility from equation (19) to the utility in equation (20), the value of longevity pooling δ_0 (in-the-large), when the individual has no pre-existing annuity income, will solve the following equation:

$$\frac{((1+\delta_0)(r+\lambda/\gamma))^{1-\gamma}}{r+\lambda/\gamma} = \frac{(r+\lambda)^{1-\gamma}}{r+\lambda}, \tag{21}$$

which after logarithms, cancelations and simplifications leads to one of our main (closed-form) expressions:

$$\delta_0 = \left(\frac{r+\lambda/\gamma}{r+\lambda}\right)^{\gamma/(1-\gamma)} - 1, \quad \text{when } \lambda > 0,\ \pi = 0 \tag{22}$$

under any combination of r, λ, γ assuming $\gamma \neq 1$; otherwise one has to take limits as $\gamma \to 1$.

To be clear, equation (22) only applies to the case in which the future remaining lifetime is assumed to be exponentially distributed with expected remaining lifetime $1/\lambda$, and in addition there is no pre-existing pension income, so that $\pi = 0$. Nevetheless, the structure of equation (22), will continue to make an appearance in all our expressions for the value of longevity risk pooling. It will have the same (ratio) format, regardless of the specific (continuous or discrete) law of mortality.

The value of longevity risk pooling in equation (22) increases in the mortality rate λ (and declines in the remaining life expectancy), but declines in the interest rate r. This characteristic of δ generalizes to non-decreasing laws of mortality, such as Gompertz-Makeham. As far as exponentially mortality is concerned, the derivative of δ_0 can be written as:

$$\frac{\partial \delta_0}{\partial r} = \frac{\gamma}{1-\gamma} \left(\frac{\lambda(1 - 1/\gamma)}{(r+\lambda)^2} \right)^{\gamma/(1-\gamma)-1}, \tag{23}$$

which is negative for $\gamma > 1$. Indeed, when interest rates are relatively lower (i.e. $r = 2\%$ vs. $r = 4\%$) the value of longevity pooling is greater to the consumer. And, the older you are (i.e. larger λ) the more you benefit from pooling. All this according to equation (23).

Note also that when the mortality rate λ happens to be such that $r\gamma = \lambda$, and the ratio of interest rates to mortality rates happens to equal the coefficient of relative risk aversion, the value of longevity pooling reported in equation (22), can be pushed further to yield:

$$\delta_0 = \left(\frac{2}{1+\gamma} \right)^{\gamma/(1-\gamma)} - 1, \quad \text{when } \lambda = r\gamma,\ \pi = 0. \tag{24}$$

This (oddly enough) does not depend on λ or the interest rate r. For example, when $\gamma = 2$ the value of longevity risk pooling is then $\delta_0 = 125\%$ in equation (24), which means that the AEW is \$2.25 per initial \$1 of wealth in the absence of any pre-existing pension income. And, when $\gamma = 1.25$ the value of longevity risk pooling is $\delta = 80\%$ in equation (24), both in the simplified case in which $\lambda/r = \gamma$. For comparison purposes, Kotlikoff and Spivak (1981) on page #378, report values for $\gamma = 1.25$ which in their case are: $\delta_0 = 97\%$ for males and $\delta_0 = 85\%$ for females at the age of 75, assuming $\rho = r = 1\%$. So these numbers – under exponential remaining lifetime assumptions – aren't very far from numbers based on more realistic mortality tables.

Interestingly, if we compute the limit of $(2/(1+\gamma))^{\gamma/(1-\gamma)} - 1$ as γ goes to 1 and the utility function converges to logarithmic, the value of longevity risk pooling in equation (24), converges to:

$$\delta_0 = \sqrt{e} - 1 \approx 64.9\%, \qquad \text{when } r = \lambda, \gamma \to 1, \tag{25}$$

which just to be clear, assumes that the mortality rate $\lambda = r$. Perhaps this is a stretch, but an interesting expression nevertheless which provides an *actuarial economic* interpretation to $\sqrt{e}$.

Now, these simple expressions are nice and helpful at the extremes of all-or-nothing annuities, but what is the value of longevity pooling δ_ψ when the individual has pre-existing pension income? We now move on to maximal utility for the generalized endowment (w, π), to obtain an expression that can be compared against equation (22). Obviously it won't be as clean and will also involve initial w as well as pension income π. More importantly, the value of δ_ψ for $\psi > 0$ will not be as large as δ_0, even when the discounted economic value of the initial endowments (w, π) are identical.

3.2 Wealth Depletion Time

The first order of business is to derive the wealth depletion time (WDT) denoted by τ, which is instrumental and a key milestone for computing the maximal utility value $U^*(w, \pi)$. We start with the basic budget constraint which dictates that:

$$w \;=\; \int_0^\tau (c_t^* - \pi)e^{-rt}dt \;=\; \pi \int_0^\tau \left(\frac{e^{-\lambda t/\gamma}}{e^{-\lambda\tau/\gamma}} - 1 \right) e^{-rt}dt. \tag{26}$$

Note that the transition from the c_t^* to the expression in the second integrand comes directly from equation (11) and the boundary condition (at the WDT) which states: $c_\tau^* = \pi$.

We are solving for τ in the above equation, which after a bit of calculus leads to:

$$\left(\frac{r}{r+\lambda/\gamma} \right) e^{\frac{\lambda}{\gamma}\tau} \;+\; \left(\frac{\lambda/\gamma}{r+\lambda/\gamma} \right) e^{-r\tau} \;=\; \frac{rw}{\pi} + 1, \tag{27}$$

where the variable τ, appears in two different exponents and is difficult to extract in closed form. It can however be solved iteratively starting with $\tau = 0$ and incrementing until reaching the value of the right-hand side. For example, when $w = 100, \pi = 10, r = 0.03, \gamma = 2$ and the mortality rate $\lambda = 1/20$, the $\tau = 28.24$ years. But, if the pension is doubled to $\pi = 20$ units, the WDT drops to $\tau = 20.08$ years. Note that for positive values of γ, λ, r, the left-hand side of the above equation is monotonically increasing in $\tau \geq 0$, which lends itself nicely to numerical methods for locating the unique value of τ for which the left-hand side is equal to $\frac{rw}{\pi}$. When $\tau = 0$, the value of the left-hand side is 1 and zero can only be a wealth depletion time if-and-only-if $w = 0$, whenever $\pi > 0$.

Finally, recall that $\psi = w/(w + a_x\pi)$, which in the case of exponential mortality, $a_x = (r+\lambda)^{-1}$, implies that equation (27) can also be written as:

$$\left(\frac{r}{r+\lambda/\gamma} \right) e^{\frac{\lambda}{\gamma}\tau} \;+\; \left(\frac{\lambda/\gamma}{r+\lambda/\gamma} \right) e^{-r\tau} \;=\; \frac{r/\psi + \lambda}{r+\lambda}.$$

This equation in τ collapses to: $re^{-\lambda\tau} + \lambda e^{-r\tau} = r/\psi + \lambda$, when $\gamma = 1$.

3.3 Special Case Again: $\lambda/\gamma = r$

When the mortality rate $\lambda = \gamma r$, the left hand side of equation (27) collapses to: $\frac{1}{2}\left(e^{r\tau} + e^{-r\tau}\right) = \cosh(r\tau)$, which is a trigonometric function, the wealth depletion time (WDT) under exponential mortality is:

$$
\begin{aligned}
\tau &= \frac{1}{r}\ln\left[\frac{rw}{\pi}+1+\sqrt{\left(\frac{rw}{\pi}+1\right)^2-1}\,\right], \quad \text{when } \lambda/\gamma = r. \\
&= \frac{1}{r}\ln\left[\frac{r/\psi+\lambda}{r+\lambda}+\sqrt{\left(\frac{r/\psi+\lambda}{r+\lambda}\right)^2-1}\,\right]. \qquad (28)
\end{aligned}
$$

The WDT τ doesn't depend on risk aversion γ or the mortality rate λ, as long as $\lambda/\gamma = r$. For example, with $\pi = 3$, $w = 60$ and $r = 2.5\%$, the value of $rw/\pi + 1 = 1.5$, which results in $\tau = \ln(1.5 + \sqrt{1.25}))/0.025 = 38.5$ years. If indeed the individual reaches this age he/she would exhaust all wealth and live-off the \$3 of pension. The consumption rate up to WDT is: $c_t = (3/(e^{(0.025)(38.5)})^{0.025t}$, according to equation (11), which is equal to: $c_t^* = 3$, when $\tau = 38.5$.

Now that we have the wealth depletion time τ, either explicitly from equation (28), or implicitly from equation (27), as well as the optimal consumption function c_t^*, we move-on to compute maximal utility.

3.4 Maximal Utility

Recall that when the mortality rate λ is constant, the immediate annuity factor can be expressed as $a_x = 1/(r+\lambda)$ and the deferred annuity factor is $a_x(\tau) = e^{-(r+\lambda)\tau}/(r+\lambda)$. The maximal value of utility, going back to the original formulation in equation (9), is:

$$
U_\lambda^*(w,\pi) = \frac{\pi^{1-\gamma}}{1-\gamma}\left[\int_0^\tau e^{-(r+\lambda)t}\left(\frac{e^{-\lambda t/\gamma}}{e^{-\lambda\tau/\gamma}}\right)^{1-\gamma} dt + \frac{e^{-(r+\lambda)\tau}}{r+\lambda}\right], \qquad (29)
$$

where the optimal consumption function comes from equation (11). Note that τ appears three times in the above expression, once in the upper bound of integration, then in the denominator of the integrand and finally in the deferred annuity factor. Once again, after a bit of calculus, equation (29) can be re-written as:

$$
U_\lambda^*(w,\pi) = \frac{\pi^{1-\gamma}}{1-\gamma}\left(\frac{e^{2r\tau}-1}{2r}+\frac{1}{r+\lambda}\right)\left(\frac{1}{e^{r\tau}}\right)^{\frac{\lambda}{r}+1}. \qquad (30)
$$

The value of longevity risk pooling *in-the-large* or *in-the-small*, will be the value of δ_ψ and v that respectively solve:

$$
U_\lambda^*(w(1+\delta_\psi),\pi) = U_\lambda^*(0,\pi+w(r+\lambda)) \qquad (31)
$$

and

$$
U_\lambda^*(w+v,\pi) = U_\lambda^*(w-1,\pi+r+\lambda). \qquad (32)
$$

With all the analytics in hand, we are now ready for some numerical examples.

3.5 Numerical Examples

Table #1 contains detailed results from two separate parametric examples. As stated in the objective, our annuity equivalent wealth (AEW) computations do not involve any dynamic programing algorithms, discretization or approximation schemes and can be easily replicated or reapplied to other parameters.

Both cases (A and B) in table #1 are based on an economy in which the (risk-free) interest rate is 2.5%, a retiree (who we call Xi) has an initial economic endowment of $100 and rational preferences that can be described using a constant relative risk aversion (CRRA) utility function. To be very clear, the $100 captures the sum of two terms: (i.) liquid non-annuitized investable wealth, and (ii.) pre-pensionized income. The rows in table #1 model different combinations of pension income plus investable wealth, but all are associated with an actuarial present value of exactly $100. The economic equivalence is important when we illustrate or compare the utility value of pooling. The Pensionized column (#2) is then also 100ψ, using our prior definition.

Let's start at the very top of Case A, when the retiree (Xi) has $100 in liquid wealth but no pre-existing pension income, which is why (in the second column) the actuarial present value of any pre-existing pension income is zero. For this initial endowment, the optimal consumption strategy for Xi is spending 5 dollars (in the first year) and rationally reduce consumption by $r + \lambda/\gamma = 5\%$ each year, as per the optimal consumption function and instructions in equation (12). Xi will never exhaust or deplete the investable funds (since $5e^{-0.05*t}$ is always positive) regardless of how long Xi lives.

To our main point, let's compare Xi to his or her utility twin (called Yi) who uses the entire 100 dollars to voluntarily purchase the (actuarially fair, unloaded) life annuity at time zero. By fully annuitizing, Yi's discounted utility of lifetime consumption would be higher. Indeed, this is the foundation of annuity (and much of pension) economics and the famous Yaari (1965) result. Nobody can beat Yi.

Back to utility. Xi would require an additional 125% of initial wealth – that is a total of 225 dollars – to be as well off (or happy) in utility terms as Yi who used the entire $100 to purchase a fairly priced life annuity. For the record, case A assumes a mortality rate of 5%, which implies that $100 would entitle Yi to 100(0.025+0.05) = 7.5 dollars of annuity income for life. Yi would immediately consume more and experience more discounted utility. A win-win situation, which isn't observed in all cases.

For purposes of replication, the maximal utility (for Yi) if the entire $100 were annuitized is: -1.777 utiles according to equation (19), whereas the maximal utility (for Xi) is exactly -4.0 utiles. Stated differently, giving Xi who has $100 of investable wealth and no annuity income an additional 125 dollars will increase discounted utility from -4 to -1.777, which is the classic definition of annuity equivalent wealth (AEW). The compensating number can be expressed as a percentage (125%), or on a per dollar basis (2.25). A pension actuary would say that the value of longevity pooling is 125% of retirement wealth, and we adopt that language.

So far there is nothing new here, although some readers might be surprised at the rel-

Table #1					
What is the Utility Value of (More) Longevity Pooling					
AEW in the large: $U^*_\lambda(w(1+\delta),\pi) \ = \ U^*_\lambda(0,\pi+w/a_x)$: **Solve for** δ**.**					
AEW in the small: $U^*_\lambda(w+v,\pi) \ = \ U^*_\lambda(w-1,\pi+1/a_x)$: **Solve for** v**.**					
Case A: Interest Rate $\rho = r = 2.5\%$, CRRA $\gamma = 2.0$, Mortality Rate $\lambda = 5.0\%$.					
Endowment ($)	**Pensionized ($)**	**Depletion**	**Consume ($)**	v **($)**	δ **(%)**
A1. $w = 100$, $\pi = 0.0$	$PV(0.000) = 0.0$	$\tau = \infty$	$c_0 = 5.000$	1.986	**125.0%**
A2. $w = 86\frac{2}{3}$, $\pi = 1.0$	$PV(1.000) = 13\frac{1}{3}$	$\tau = 72.8$ yrs.	$c_0 = 6.171$	1.668	**114.8%**
A3. $w = 73\frac{1}{3}$, $\pi = 2.0$	$PV(2.000) = 26\frac{2}{3}$	$\tau = 50.7$ yrs.	$c_0 = 7.104$	1.432	**104.2%**
A4. $w = 60$, $\pi = 3.0$	$PV(3.000) = 40$	$\tau = 38.5$ yrs.	$c_0 = 7.854$	1.232	**93.0%**
A5. $w = 46\frac{2}{3}$, $\pi = 4.0$	$PV(4.000) = 53\frac{1}{3}$	$\tau = 29.8$ yrs.	$c_0 = 8.437$	1.049	**80.9%**
A6. $w = 25$, $\pi = 5.625$	$PV(5.625) = 75$	$\tau = 18.6$ yrs.	$c_0 = 8.974$	0.743	**57.7%**
A7. $w = 10$, $\pi = 6.750$	$PV(6.750) = 90$	$\tau = 10.9$ yrs.	$c_0 = 8.854$	0.468	**35.7%**
A8. $w = 1$, $\pi = 7.425$	$PV(7.425) = 99$	$\tau = 3.28$ yrs.	$c_0 = 8.060$	0.110	**11.0%**
A9. $w = 0$, $\pi = 7.500$	$PV(7.500) = 100$	$\tau = 0$	$c_0 = 7.500$	N.A.	N.A.
Case B: Interest Rate $\rho = r = 2.5\%$, CRRA $\gamma = 1.25$, Mortality Rate $\lambda = 3.125\%$.					
Endowment ($)	**Pensionized ($)**	**Depletion**	**Consume ($)**	v **($)**	δ **(%)**
B1. $w = 100$, $\pi = 0.0$	$PV(0.000) = 0.00$	$\tau = \infty$	$c_0 = 5.000$	1.243	**80.2%**
B2. $w = 82.23$, $\pi = 1.0$	$PV(1.000) = 17.77$	$\tau = 71.3$ yrs.	$c_0 = 5.943$	1.035	**72.0%**
B3. $w = 64.45$, $\pi = 2.0$	$PV(2.000) = 35.55$	$\tau = 47.9$ yrs.	$c_0 = 6.618$	0.869	**63.2%**
B4. $w = 46.67$, $\pi = 3.0$	$PV(3.000) = 53.33$	$\tau = 34.2$ yrs.	$c_0 = 7.058$	0.716	**53.4%**
B5. $w = 28.89$, $\pi = 4.0$	$PV(4.000) = 71.11$	$\tau = 23.7$ yrs.	$c_0 = 7.232$	0.555	**41.8%**
B6. $w = 10$, $\pi = 5.063$	$PV(5.063) = 90.00$	$\tau = 12.5$ yrs.	$c_0 = 6.923$	0.330	**24.6%**
B7. $w = 1$, $\pi = 5.568$	$PV(5.568) = 99.00$	$\tau = 3.79$ yrs.	$c_0 = 6.122$	0.078	**7.8%**
B8. $w = 0$, $\pi = 7.625$	$PV(5.625) = 100.0$	$\tau = 0$	$c_0 = 5.625$	N.A.	N.A.
Assumes the Remaining Lifetime (T_x) is Exponentially Distributed with Mortality Rate λ. The Expected Remaining Lifetime and Standard Deviation of Lifetime (T_x) are both $1/\lambda$ years. The value of a $1-per-year life annuity is: $a_x = 1/(r+\lambda)$ when (T_x) is Exponentially Distributed.					

atively high value of the AEW (or pooling), compared to results reported in Brown et al (2001) for example. They use a similar 2.5% interest rates and obtain pooling values of (only) 65%, when the coefficient of relative risk aversion of the representative consumer is equal to 2.0, which is what we used here.

The main explanation for our higher AEW or pooling values is that we have assumed an exponentially distributed remaining lifetime, with a mean value of 20 years and a standard deviation of 20 years. We really had no choice since it is forced by the 5% mortality rate underlying the exponential remaining lifetime. This (for example) leads to a 10% probability of spending 45 years in retirement and living beyond age 110. This mortality assumption does result in a much higher value for the longevity pooling, which is why our numbers are higher than Brown et al. (2001).

Note that they used projections for the 1930s Social Security Administration cohort for which life expectancy was 15 years at the age of retirement. In some sense, our exponential remaining lifetime assumptions are biased in favor of higher AEW and pooling values due to the *fatter longevity tail* of the exponential distribution. But our rationale in this section was to illustrate how these values decline when pre-existing annuities are included in the initial endowment. In the next section (#4) we will display AEW numbers and the value of pooling for more realistic mortality assumptions. There, we aren't as fortunate to have analytic expressions for all quantities of interest, but the numbers (better) match those in Brown et al. (2001). See the technical appendix for R-script code that generates these values.

Moving on, the second to last column in table #1 displays the results of the annuity equivalent wealth (AEW) *in-the-small.* The number is intended to answer the question: How much additional wealth would Xi the retiree (who recall has no annuity income, but \$100 in liquid wealth) require to be as satisfied as the utility twin Yi who only converted \$1 into annuity income? The hypothetical twin doesn't convert the entire \$100 dollars in annuities, only 1 dollar. We are focusing on the marginal impact. Our numbers are based on the v that satisfies equation (32).

To be clear, we are now comparing the utility of someone (Yi) who has \$99 of liquid wealth plus (0.025+0.05)=0.075 dollars in lifetime income, to someone (Xi) who has \$100 in liquid wealth and absolutely no pension or annuity income. The actuarial present value of the 0.075 dollars of annuity income is equal to 1 dollar, so the initial economic endowment of both twins remains the same.

According to equation (32), Xi would now require or demand an additional \$1.986 (for a total wealth of \$101.986) to obtain the same level of discounted lifetime utility, compared to twin Yi with \$99 of liquid wealth and \$0.075 of annual annuity income. Notice that as a percent of the one dollar used by Yi to purchased annuities, the \$1.986 is much more than the 125%, which we associated with annuity equivalent wealth *in-the-large.*

The reason for this apparent non-homogeneity, despite the assumed CRRA utility structure, is that (even) when a small sum is converted into annuity income, the discounted value of lifetime utility, or the value function no longer scales in initial wealth. (It does scale in the pension income, though.) This is because of the wealth depletion time which is an integral (albeit obscured) part of the optimal consumption strategy. The sixth row (Case A6) illustrates a consumer (still Xi) who is entitled to a pension of \$5.625 for life, but who has liquid wealth (savings) of \$25 at retirement. Note that the multiple of pension income to investable wealth is 4.4 to one, which is consistent with a typical (American) retiree who is entitled to social security benefits of \$32,000 per year and might have (an average of) \$140,000 in retirement savings. Within table #1, this retiree also has an initial \$100 dollars, but 75% is pre-allocated to pension annuities, only 25% is available for further annuitization; $\psi = 0.75$.

The extra utility from annuitizing the remaining \$25 will be lower, and the last column in table #1 indicates that the value of pooling for this situation is (only) 57.7%. This number is less than half of the value of pooling when the individual has no pre-existing annuity income (recall the 125%), but it still quite substantial. To put this number in a sentence, the annuity equivalent wealth of \$25 in investable wealth when you have \$5.625 in pension income is $25(1.577) = \$39.425$. This once again is what we term the value of longevity risk pooling *in-the-large.*

The second to last column in the same row indicates that the parallel AEW *in-the-small* is 0.743 dollars. This means that Xi with 25 dollars would require an additional 0.743 dollars (or 25.743) to make them as well-off, in a utility sense, as Yi who has 24 dollars of investable wealth and $5.625 + 0.075 = 5.7$ dollars of annuity income. Either way, whether we focus on the absolutely (large) or marginal (small) the value of longevity risk pooling is non-zero, but getting lower and closer to the insurance loadings (and anti-selection costs) that one might observe in practice. More on this in the conclusion.

Moving on to the optimal consumption strategy for this individual (\$25 of investable wealth and \$5.625 of annuity income), the wealth depletion time (WDT) is 18.6 years. As we explained in the earlier section, Xi will rationally exhaust wealth after 18.6 years of retirement and (from that point onward) only consume the pension income of \$5.625. The initial consumption rate will be \$8.437 per year, of which \$5.625 is pre-existing pension income and \$2.812 comes from liquid investable wealth. Using terminology common among financial advisors, this is a drawdown or retirement spending rate of 2.812/25 or 11%, and fully rational given the (higher) pension income.

At the very bottom of the Case A section, notice that when 99% of the retiree's initial endowment is already pre-annuitized, both the AEW *in-the-large* and the AEW *in-the-small* is equal to 11%. Their definitions now coincide. Giving-up one dollar (and buying more annuities) leads to the same utility as annuitizing the entire one dollar.

Table #1 also offers numbers (under Case B) when the mortality rate λ and the coefficient of relative risk aversion γ are reduced. For the second case, the CRRA value is set at $\gamma = 1.5$ and the mortality rate is set to $\lambda = 3.125\%$, which is synonymous with a life expectancy of 32 years (versus the 20 years in Case A.) We selected these precise numbers for Case B, because we wanted to take advantage of the complete analytic representation for τ and δ, that only work when $\lambda/\gamma = r$.

A lower or reduced value for the coefficient of relative risk aversion γ results in a lower value of longevity risk pooling δ_0, δ_π or v. Note also that optimal consumption c_t^* is reduced or lower when γ is higher, due to the higher (risk adjusted) probability of living to an advanced age. As in Case A which was reported above, all of these numbers were obtained using the analytic expressions in the prior section, although the value of v within equation (32) was extracted using a simple root-finding algorithm.

Before we conclude this numerical section (under exponential lifetimes) and move on to more realistic mortality tables and curves, it's worth commenting on the non-monotonic level of consumption as one goes down the rows in the two cases. The initial consumption rate in the absence of any annuities is \$5 per year for life, but then increases as the fraction of pre-existing pension income is increased. Eventually it does reach a maximum when 70% to 75% of the initial endowment is pre-pensionized. It then begins to decline with additional annuitization. This shouldn't be viewed as troubling or inconsistent. A higher initial consumption rate doesn't necessarily lead to additional utility primarily because consumption is forced to decline (more) rapidly over time. Stated differently and perhaps counter-intuitively, maximizing discounted lifetime utility doesn't necessarily maximize the initial consumption or spending rate.

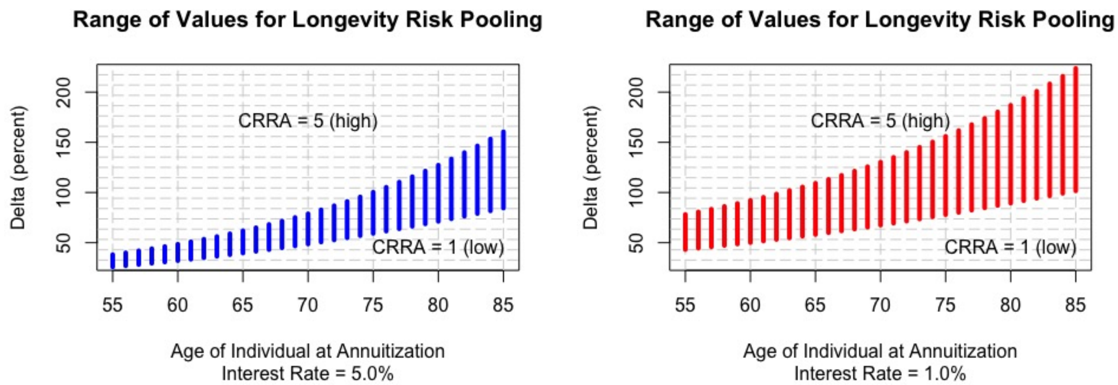
That said, within the context of behavioral economics it might be hard to convince Xi who is rationally and optimally spending \$8.974 per year (sixth row of Case A, for example) that they should annuitize the \$25 of liquid wealth (like Yi), so that their consumption rate can immediately drop to \$7.5 per year. Xi might not be persuaded that such an action is equivalent (in utility terms) to having 57.7% more liquid wealth. This is quite different from (2nd row, Case A) where the individual who is persuaded to annuitize his or her entire wealth can immediately benefit from a jump in consumption from \$6.171 per year to \$7.5 dollars per year, for life. Alas, perhaps it is not surprising that (in the real world) annuitization can be a hard sell.

In the next section #4, we offer some analytic insights for the case in which remaining lifetime is no longer exponentially distributed and the force of mortality increases over time and/or is fit to a proper life table.

4 General Mortality Models

We do not want to get bogged down in the details of an appropriate model for current and future mortality rates λ_t and instead refer the readers to the book by Pitacco, Denuit, Haberman and Olivieri (2009) for appropriate analytic representations. In this section we continue to use the notation a_x to represent the general annuity factor under an interest rate r, assuming the current age of the individual is x. The annuity factor can actually be expressed analytically when the underlying mortality basis is assumed to obey a Gompertz-Makeham law, and this is presented in the book by Charupat et al. (2012), page #289, for example. Alternatively, under a discrete mortality table (with or without projection scales), the expression a_x is easily available and computed using any basic spreadsheet. Either way, using the same methodology we described and followed in section #3, when $\pi = 0$, one obtains the following analytic expression for the value of longevity risk pooling:

Figure 1: *Displayed δ_0 values are function of initial annuitization age x, risk aversion γ and interest rate. Blue (left figure) is for $r = 5\%$, red (right figure) is for $r = 1\%$, both with risk aversion coefficients $\gamma = 1$ (lower) to $\gamma = 5$ (upper). Mortality is fit to 1930 SSA cohort.*



$$\delta_0 \;=\; \left(\frac{a_x}{a_{x^*}}\right)^{\frac{\gamma}{1-\gamma}} - 1, \tag{33}$$

where the (new) variable x^* in the denominator of equation (33) represents a modified or risk-adjusted age, similar to the reduction of the mortality rate λ/γ that appeared earlier in equation (22). One can think of x^* as an age set-back. In particular, if we assume Gompertz mortality with parameters (m, b), the adjusted age $x^* = x - b \ln \gamma$, where γ is the coefficient of relative risk aversion. For example, if $\gamma = 2$, the current age $x = 65$ and the dispersion coefficient $b = 11.5$, (which is approximately the standard deviation of the remaining lifetime), then the risk-adjusted age $x^* = 57.03$. But if the risk aversion coefficient is increased to $\gamma = 3$, then $x^* = 52.376$.

Here is another way to express and think about this. The annuity factor in the numerator of the fraction in equation (33) is computed using the retiree's **biological** age, whereas the denominator is computed at a risk-adjusted **economic** age.

4.1 In a Gompertz-Makeham mortality world

The general methodology for computing δ_0, captured by equation (33) is applicable under any mortality basis in which the implied force of mortality is non-decreasing and follows from equation (16). The derivation or proof is relatively simple in the context of Gompertz-Makeham (GM) mortality. In particular, the justification for the age-shift from x to x^* in the annuity factor, is as follows. Recall that the optimal consumption function c^*_t can be written in terms of either the initial consumption rate c^*_0 or the terminal (pension) consumption rate π, via the adjusted survival probability $({}_tp_x)^{1/\gamma}$. This comes directly from equation (12) or equation (11). Those analytic expressions for consumption are exponential in t, easily flow through the integral calculations, which lead to the value of longevity risk pooling.

In the case of GM mortality the risk-adjusted survival probability can be expressed as:

$$({}_tp_x)^{1/\gamma} \;=\; e^{-\frac{1}{\gamma}\int_0^t \lambda_s ds}, \tag{34}$$

Table #2		
Comparing Analytics to Numerics: Annuity Equivalent Wealth		
CRRA	**Gompertz Mortality (analytic)**	**Results in Brown et. al. (2001)**
$\gamma = 1.0$	$1 + \delta_0 = 1.499$	1.502
$\gamma = 2.0$	$1 + \delta_0 = 1.650$	1.650
$\gamma = 5.0$	$1 + \delta_0 = 1.872$	1.855
$\gamma = 10$	$1 + \delta_0 = 2.050$	2.004
Gompertz (Unisex) Mortality with: $m = 81, b = 11.5$ and $r = \rho = 2.5\%$ discount rate.		
Life Expectancy of 15.4 years at age 65, based on the 1930s SSA cohort.		

where the GM mortality hazard rate can be written as:

$$\lambda_s \;=\; \lambda + \frac{1}{b}\exp\left(\frac{x+s-m}{b}\right). \tag{35}$$

If we divide both sides and scale by γ, the relevant integrand can be expressed as:

$$\frac{\lambda_s}{\gamma} \;=\; \frac{\lambda}{\gamma} + \frac{1}{b}\exp\left(\frac{x - b\ln\gamma + s - m}{b}\right) \;=\; \frac{\lambda}{\gamma} + \frac{1}{b}\exp\left(\frac{x+s-(m+b\ln\gamma)}{b}\right) \tag{36}$$

where λ is replaced with λ/γ and the initial age x is replaced with $x^* = x - b\ln\gamma$. Alternatively, the initial age remains x and the modal parameter m is replaced with $m^* = m + b\ln\gamma$.

More generally, for those who want to compute the value of δ_0 for any given discrete mortality table, the trick is to adjust the underlying mortality rate by the coefficient of relative risk aversion γ, which would (roughly) approximate the process of computing λ/γ. If more precision is desired, then it would probably be best to locate the best fitting biometric (λ, m, b) values and then use the analytic Gompertz-Makeham annuity factor with the adjusted modal value: $m + b\ln\gamma$ and with λ/γ. See technical appendix for an algorithm.

4.2 Results: Continuous Gompertz vs. Discrete Brown

Table #2 compares results for the annuity equivalent wealth computed in the paper by Brown et al. (2001), vs. the value as per equation (33) and a Gompertz specification with $m = 81$ and $b = 11.5$. The two coefficients were selected to (best) fit the discrete mortality table used in Brown et al (2001). The interest and valuation rate was assumed to be 2.5%, in real terms. We report value of $1 + \delta_0$ under a variety of γ values.

The numbers in the first column of this table roughly match the values computed by Brown, Mitchell, Poterba and Warshawsky (2001), pg. 143, using mortality data from the Social Security Administration (SSA) for the 1930s cohort. For example, if the individual is extremely ($\gamma = 10$) risk averse, he/she would require 2.05 to achieve the same (maximal) level of utility as someone who spends \$1.00 and uses the funds to acquire annuity income $1/a_{65}$. The numbers reported in Brown et al. (2001) are \$2.004 per initial \$1.

Finally, figure #1 provides a full spectrum of results for different retirement (pooling) ages (55 to 85), interest rates ($r = 1\%$ and $r = 5\%$), and levels of risk aversion ($\gamma = 1$ to $\gamma = 5$.) Notice how the sensitivity to risk aversion γ increases with retirement age.

4.3 Discrete Mortality: Gender Specific

Here is another example using a different set of mortality tables and rates. Under the Individual Annuity Mortality (IAM) 1983 (basic) table, which was also used in the literature for many of the original AEW estimates, and an $r = 3\%$ interest rate, the annuity factor at age $x = 65$ is $a_{65} = 13.64645$ for males and $a_{65} = 15.58935$ for females. Assuming a $\gamma = 2$ coefficient of risk aversion in a CRRA utility function and a subjective discount rate ρ equal to the interest rate r, the modified annuity factor is $a_{65^*} = 16.81724$ for males and $a_{65^*} = 18.39907$ for females. For this we have kept it simple and divided the mortality rates in the IAM1983 table by $\gamma = 2$ and priced the appropriate (modified) annuity at age $x = 65$ using the scaled mortality. The value of annuity equivalent wealth (AEW) which is δ, is equal to: $\delta_0 = \left(\frac{13.64645}{16.81724}\right)^{-2} - 1 = 0.5187$, or 51.87% for a male at age 65, and the equivalent value is $\delta = 0.3930$ or 39.30% for a female at age 65. The AEW is lower for females because their mortality rate is lower at all ages, making the annuity (pooling) relatively more expensive, etc.

Up until now we have focused on values of $\gamma \geq 1$, but even when $\gamma < 1$ there is still value to longevity risk pooling (although it will obviously be lower), provided that $\gamma > 0$. So, if we reduce the (longevity) risk aversion parameter from 2 (in the prior paragraph) to $\gamma = 0.50$, the corresponding (objective) annuity factors a_{65} do not change. But, the modified or adjusted factors are now (increased) to $a_{65^*} = 10.53740$ for males and $a_{65^*} = 12.72198$ for females. The corresponding values of δ are (only) 29.5% for males and 22.54% for females, according to equation (33). Stated differently, the value of pooling is reduced when you don't *dislike* longevity risk as much.

For the third and final numerical example, we leave the coefficient of (longevity) risk aversion at $\gamma = 0.5$, but reduce the valuation (or pricing) interest rate from 3% to $r = 1.5\%$, which increases a_{65} as well as a_{65^*}. The value of longevity risk pooling is now (slightly higher than the prior numerical example) at 33.93% for males and 26.39% for females.

Note that the results are not as sensitive to interest rates. The δ benefits really are driven by risk aversion γ and by the assumed mortality rates. Additional numerical examples are reported in a technical appendix, with an algorithm that can also be used when $\psi > 0$.

5 Conclusion

Against the backdrop of declining defined benefit (DB) pension coverage and increasing reliance on defined contribution (DC) investment plans, there is a growing awareness that *longevity risk pooling* is being lost in transition. Practicing actuaries, as well as insurance and pension economists are generally aware of the benefits of longevity risk pooling, but the general public, the media and often regulators do not appreciate the social welfare benefits of annuitization.

In this paper we take a step in the direction of helping *explain* and quantify the benefits of pooling to practising actuaries and a wider public by deriving some closed-form and easily digestible expressions for the utility-based value of longevity pooling. This utility value is simply the percentage increase in what economists label *annuity equivalent wealth* (AEW). The algorithmic process for computing AEW value has been used by economists in the literature, going back to the work by Kotlikoff and Spivak (1981), Brown et al. (2001), or the reference on annuity markets by Cannon and Tonks (2008). What is less known – and we believe one of the contributions of this paper – is that by assuming some analytic representations for the remaining lifetime random variable, one can obtain closed-form expressions for the value of longevity pooling. These formulae are simply unavailable or highly obscured when the mortality basis is a discrete mortality table and the optimization is done via backward induction or numerical dynamic programming.

Under logarithmic utility preferences and the assumption that remaining lifetime is exponentially distributed with a mean value of: $1/\lambda$ (for example 20 years) *and* assuming the risk-free interest rate is coincidentally equal to the instantaneous mortality rate (for example 2.5%), then the value of longevity risk pooling is: $\sqrt{e}-1$, which is approximately 65%. Note that in Brown et al. (2001), using a dynamic programing algorithm and a variety of mortality tables and interest rate assumptions, the reported result for the AER was 1.5, or a 50% value from pooling. To be clear, our $\delta_0 = \sqrt{e}-1$ result only holds when the inverse of life expectancy (λ) is equal to the risk-free rate (r). And, while this result might be trivial or impractical or both, the fact is that it provides bounds on the value of longevity risk pooling.

More generally when the life expectancy is not equal to the interest rate, the value of longevity pooling can be expressed in an equally simplified manner as a power function of the ratio of two annuity factors. Under *any* combination of interest rates r and mortality rates λ, the value of longevity risk pooling is:

$$\delta_0 = \left(\frac{r+\lambda/\gamma}{r+\gamma}\right)^{\frac{\gamma}{1-\gamma}} - 1 := \left(\frac{a_\lambda}{a_{\lambda/\gamma}}\right)^{\frac{\gamma}{1-\gamma}} - 1.$$

The first annuity factor in the numerator is the actual one based on biological age and mortality rate: $a_\lambda = (r+\lambda)^{-1}$, and the second annuity factor in the denominator assumes an age set-back that depends on the degree of risk aversion, $a_{\lambda/\gamma} = (r+\lambda/\gamma)^{-1}$.

For example, in the case of exponential mortality a $\gamma = 2$ would imply a doubling of life expectancy from $1/\lambda$ to $2/\lambda$, etc. So, the value of longevity risk pooling at the age of 65 is equal to the actuarial annuity factor for a 65-year-old divided by the actuarial annuity factor for a 45-year-old (all to the power of 2). The retiree is 65, but the pricing is done as if they are 45, etc.

In fact, this simplified representation of the value of longevity risk pooling δ_0 extends to all continuous (biologically reasonable) mortality assumptions, which would include Gompertz-Makeham which can be fit to any (retirement) mortality table. To that end we presented expressions which can be used in many circumstances and offer some economic and actuarial intuition.

Besides novel expressions, in terms of lifecycle modeling we have shown that in the presence of pre-existing pension annuities, not only is the value of longevity pooling lower, but the methodology used must be carefully adjusted for the rational wealth depletion time (WDT). It is no longer possible to express the value of pooling as a function of a simple ratio of annuity factors and in fact the discounted value function no longer scales in initial wealth. At the very least one must scale by pension income. And, while we are still able to obtain some analytic results under exponential mortality – and reported those in table #1 – we are forced to parse the definition of pooling value to differentiate between annuitizing one more dollar vs. the entire initial wealth. In contrast to what someone might expect, the value of pooling one more dollar (v using our notation) is worth more than the value of pooling the entire wealth (δ, using our notation), even in the presence of pre-existing annuities. We referred to this as the annuity equivalent wealth *in-the-large* vs. *in-the-small*.

5.1 Final Words

We conclude with a note reminding readers that all of these results and expressions were derived assuming the biological annuity factors a_x applicable for valuing pre-existing pension annuity income are identical to the cost of buying additional annuity income. Early we mentioned that the price of \$1 of annuity income is $(1+\kappa)a_x$, but assumed $k = 0$. We placed ourselves in a utopia with no adverse selection costs and no transaction costs. Mandated groups (e.g. Social Security) pay the same price for their pension annuities as retail individuals would pay in the open market. This is not the case in practice and $\kappa > 0$. See, for example, estimates provided by Finkelstein and Poterba (2004), for the magnitude of these costs in the U.K. It's evident from the numerical results that if one already has $\psi = 75\%$ of retirement wealth pre-annuitized, and one has to pay a loading of 30% to acquire additional annuity income, possibly due to stochastic mortality, the value of pooling can be negative. The rational consumer would be better off spending-down assets and living-off pension annuity income at the wealth depletion time. We leave these frictions as an avenue for future research. At what value of $\kappa > 0$ does δ_ψ become negative and annuitization reduces utility?

Our research has also omitted any discussion of more general Epstein-Zin preferences, in which obtaining closed-form expressions is quite hopeless. Although, according to estimates reported in Cannon and Tonks (2008), the annuity equivalent wealth values would be even higher than our numbers. On the other hand recent and highly visible work by Rechling and Smetters (2015) indicates that in the presence of stochastic mortality the value of pooling is lower, although this has been questioned in recent work by others, see for example Bauer (2017). On a separate path, Feigenbaum et al. (2013) use an overlapping generation model to argue that full annuitization is not welfare maximizing even with a deterministic mortality rate. In other words, the annuity debate (in academia) continues.

In sum, we believe that building a strong case or consensus for annuitization in the future will crucially depend on a strategy of being able to explain the value to individual retirees who are limited by behavioral and cognitive obstacles. This is in contrast to a research strategy of extending the economic lifecycle model to include more general behavioral preferences, longevity insurance products and asset dynamics. In some sense this group is preaching to the (very small) choir of economists within the existing literature. We hope that some of the simple, analytic and digestible expressions we presented in this paper might be lead to contributions beyond the academic literature. At the very least $\sqrt{\epsilon}$ now has an *actuarial economic* interpretation.

References

[1] Bauer, D. (2017) Longevity risk pooling: Opportunities to increase retirement security, *working paper*, Georgia State University and Society of Actuaries

[2] Blake, D. (1999), Annuity markets: Problems and Solutions, *Geneva Papers on Risk and Insurance*, Vol. 24(3), pg. 358-375

[3] Bodie, Z. (1990), Pensions as retirement income insurance, *Journal of Economic Literature*, Vol. 28, pg. 28-49

[4] Bommier, A. (2006), Uncertain lifetime and inter-temporal choice: risk aversion as a rationale for time discounting, *International Economic Review*, Vol. 47, pp. 1223-1246.

[5] Brown, J.R. (2001), Private pensions, mortality risk and the decision to annuitize, *Journal of Public Economics*, Vol. 82, pg. 29-62.

[6] Brown, J. R., O.S. Mitchell, J.M. Poterba, and M.J. Warshawsky (2001), *The Role of Annuity Markets in Financing Retirement*, MIT Press, Cambridge.

[7] Brown, J.R., J.R. Kling, S. Mullainathan and M.V. Wrobel (2008), Why don't people insure late-life consumption? A framing explanation of the under-annuitization puzzle, *American Economic Review*, Vol. 98(2), 304-309.

[8] Butler, M. (2001), Neoclassical life-cycle consumption: a textbook example, *Economic Theory*, Vol. 17, pp. 209-221.

[9] Cannon, E. and I. Tonks (2008), *Annuity Markets*, Oxford University Press, New York City, NY.

[10] Charupat, N., H. Huang and M.A. Milevsky (2012), *Strategic Financial Planning over the Lifecycle: A Conceptual Approach to Personal Risk Management*, Cambridge University Press, New York, NY.

[11] Cocco, J.F. and F.J Gomes (2012), Longevity risk, retirement savings and financial innovation, *Journal of Financial Economics*, Vol. 103(3), pp. 507-529

[12] Davidoff, T., J.R. Brown, P.A. Diamond (2005), Annuities and individual welfare, *American Economic Review*, Vol. 95(5), pg. 1573-1590.

[13] Davies, J.B. (1981), Uncertain lifetime, consumption and dissaving in retirement, *Journal of Political Economy*, Vol. 89, pp. 561-577.

[14] Feigenbaum, J. A., E. Gahramanov, and X. Tang, (2013), Is It Really Good to Annuitize? *Journal of Economic Behavior and Organization*, Vol. 93, pg. 116-140.

[15] Finkelstein, A. and J. Poterba (2004), Adverse selection in insurance markets: Policyholder evidence from the U.K. annuity market, *Journal of Political Economy*, Vol. 112(1), 183-208.

[16] Horneff, W.J., R. H. Maurer, O. S. Mitchell and M. Z. Stamos (2009), Asset allocation and location over the life cycle with investment-linked survival-contingent payouts, *Journal of Banking and Finance*, Vol. 33(9), pp. 1688-1699.

[17] Kotlikof, L.J. and A. Spivak (1981), The family as an incomplete annuity market, *Journal of Political Economy*, Vol. 89(2), pg. 372-391.

[18] Lachance, M. (2012), Optimal onset and exhaustion of retirement savings in a life-cycle model, *Journal of Pension Economics and Finance*, Vol. 11(1), pp. 21-52.

[19] Leung, S. F. (2002), The dynamic effects of social security on individual consumption, wealth and welfare, *Journal of Public Economic Theory*, Vol. 4(4), pg. 581-612.

[20] Leung, S. F (2007), The existence, uniqueness and optimality of the terminal wealth depletion time in life-cycle models of saving under certain lifetime and borrowing constraint, *Journal of Economic Theory*, Vol. 134, pp. 470-493.

[21] Levhari, D. and L.J. Mirman (1977), Savings and consumption with an uncertain horizon, *Journal of Political Economy*, Vol. 85(2), pp. 265-281.

[22] Pashchenko, S. (2013), Accounting for non-annuitization, *Journal of Public Economics*, Vol. 98, pg. 53-67.

[23] Pitacco, E., M. Denuit, S. Haberman and A. Olivieri (2009), *Modelling Longevity Dynamics for Pensions and Annuity Business*, Oxford University Press, NY.

[24] Reichling, F. and K. Smetters (2015) Optimal Annuitization with Stochastic Mortality and Correlated Mortality Cost, *American Economic Review*, Vol. 11, pg. 3273-3320

[25] Sheshinski, E. (2007), *The Economic Theory of Annuities*, Princeton University Press, Princeton.

[26] Yaari, M.E. (1965), Uncertain lifetime, life insurance and the theory of the consumer, *The Review of Economic Studies*, Vol. 32(2), pp. 137-150.

Appendix & Algorithm: Utility Value of Longevity Risk Pooling

Our objective in this technical appendix is to provide some self-contained R-scripts that can be used to compute (numerically) the annuity equivalent wealth $\hat{w}$, or what we call the value of longevity risk pooling $\delta := \hat{w}/w - 1$, when the retiree is endowed with investable wealth w and pre-existing annuity income flow π. Recall that the closed-form analytic equation (33) in Milevsky & Huang (2018) only "works" when $\pi = 0$. When $\pi > 0$ the value of longevity risk pooling δ is lower than equation (33) and the solution structure itself is more complicated. Instead of the relevant δ being equal to the ratio of two annuity factors to the power of risk aversion, we must (i.) compute the wealth depletion time τ at the same time we (ii.) equate utilities. We now describe that process or algorithm, one which also reproduces equation (30) when $\pi = 0$.

5.2 Wealth Depletion Time

We start by defining the basic annuity valuation factor (or function in R), using the current age x, the valuation rate r, and the Gompertz parameters (m, b) as arguments.

```
ILA<-function(x,r,m,b){
  omega<-x+b*log(1+10*log(10)*exp((m-x)/b));
  dt<-1/52; grid<-(omega-x)/dt; t<-(1:grid)*dt
  pgrid<-exp(exp((x-m)/b)*(1-exp(t/b)))
  rgrid<-exp(-r*t)
  sum(pgrid*rgrid)*dt}
```

This *immediate* life annuity factor is a discretized (Reimann) version of an actuarial expectation, which we will occasionally abbreviate as:

$$\texttt{ILA(x)} := \int_0^{\omega - x} \exp\{-rt + e^{(x-m)/b}(1 - e^{t/b})\}\, dt, \tag{37}$$

since (r, m, b) are usually fixed. In the R-script, the grid size: $dt = 1/52$ (weekly) is arbitrary, as is the upper bound: $\omega - x = b \ln\{1 + 10 \ln(10) e^{(m-x)/b}\}$. At first glance, this cut-off point might seem arbitrary, but under a Gompertz law of mortality the conditional survival probability from age x to age ω is exactly: $({}_{\omega-x}p_x) = 10^{-10}$, under this value of ω. For example, when $m = 90$, $b = 10$ and $x = 65$, the integral's upper bound is 56.401 This represents a one-in-ten billion chance of a new retiree living to age 121.4; roughly consistent with the death of Jeanne Calment at age 122, so far the world's oldest living person. Note that our R-script construction of `ILA(x)` is general enough to accommodate any (not only Gompertz) survival curve via the argument `pgrid` and any (not only constant r) valuation curve via `rgrid`, so long as they can both be expressed in consistent weekly increments. Using the same approach, we construct a *temporary* life annuity factor with the following modified R-script.

```
TLA<-function(x,T,r,m,b){
  dt<-1/52; grid<-T/dt; t<-(1:grid)*dt
  pgrid<-exp(exp((x-m)/b)*(1-exp(t/b)))
  rgrid<-exp(-r*t)
  sum(pgrid*rgrid)*dt}
```

Lifetime payments cease at the earlier of death and $T \leq \omega - x$. The only difference between the original `ILA` and `TLA` is the time horizon T which determines the upper bound of integration instead of $\omega - x$. Computationally both factors are identical when $T = \omega - x$, and the formal actuarial definition, is:

$$\texttt{TLA}(\texttt{x},\texttt{T}) \;=\; \int_0^T \exp\{-rt + e^{(x-m)/b}(1-e^{t/b})\}\,dt. \tag{38}$$

We should note that under a Gompertz-Makeham law of mortality there are in fact closed form analytic expressions available for `TLA(x,T)` in terms of the well-known Gamma function. See Charupat et. al (2012, pg. 289) for example. Moreover, using a Gamma representation may be faster (in R) compared to the crude discretization scheme described above. We have decided to adopt the numerical (grid) approach for ease of explanation, replicability and because the Gamma function can get "finicky" under certain parameter values.

To complete the trivariate, using basic cash-flow geometry we construct the *deferred* life annuity factor by subtracting a `TLA` from the `ILA` in the following intuitive way.

```
DLA<-function(x,T,r,m,b){ILA(x,r,m,b)-TLA(x,T,r,m,b)}
```

This function could also have been constructed by subtracting two temporary annuity factors `TLA(x,T)` from `TLA(x,`ω`-x)`, both of which follow directly from the actuarial expectation:

$$\texttt{DLA}(\texttt{x},\texttt{T}) \;=\; \int_T^{\omega-x} \exp\{-rt + e^{(x-m)/b}(1-e^{t/b})\}\,dt, \tag{39}$$

As far as modifying the R-script for the `DLA` is concerned, reducing the grid size, changing ω, replacing the embedded mortality curve `pgrid` and/or discounting curve `rgrid` must be done directly via the functions `ILA(x)` and `TLA(x,T)`. For the sake of calibration or replication we offer the following three numerical values of the relevant functions.

```
> ILA(65,0.025,81,11.5)
[1] 12.21481
> TLA(65,15,0.025,81,11.5)
[1] 9.96662
> DLA(65,15,0.025,81,11.5)
[1] 2.248191
```

These numbers reflect an $r = 2.5\%$ (real) valuation rate and Gompertz parameters $m = 81, b = 11.5$, as per table #2 in Milevsky & Huang (2018), which itself was calibrated to the 1930 Social Security Administration (SSA) cohort. Moreover, using the classic actuarial identity that the immediate annuity factor collapses to mathematical life expectancy $E[T_x]$, when the interest rate is zero, recovers the (cohort) life expectancy for the 1930 SSA life table.

```
> ILA(65,0,81,11.5)
[1] 15.44889
```

We are ready to compute the wealth depletion time τ. As shown in Charupat et al. (2012, pg. 301), while the optimal consumption function $c_t^* = \pi$, for time $t \in [\tau, \infty)$, which is *after* the wealth depletion time, *during* the period $t \in [0, \tau)$, $c_t^* = c_0^*({}_tp_x)^{1/\gamma}$. This can also be expressed as:

$$c_t^* \;=\; \left[\frac{(w+\pi/r)e^{r\tau} - \pi/r}{\texttt{TLA}(\texttt{x}{-}b\ln\gamma, \tau)\, e^{r\tau}}\right]({}_tp_x)^{1/\gamma}, \;\; \forall\, t \le \tau, \tag{40}$$

where $\texttt{TLA}(\texttt{x}{-}b\ln\gamma, \tau)$ is a temporary life annuity factor, terminating at the earlier of death and τ years, but assuming the current (Biological) age is set-back to: $x - b\ln\gamma$. Equation (40) implies that the wealth depletion time τ must satisfy:

$$\left[\frac{(w+\pi/r)e^{r\tau} - \pi/r}{\texttt{TLA}(\texttt{x}{-}b\ln\gamma, \tau)\, e^{r\tau}}\right]({}_\tau p_x)^{1/\gamma} \;-\; \pi \;=\; 0, \tag{41}$$

because consumption c_t^* converges to π as $t \to \omega - x$. Note that τ appears in no less than four places within equation (41). This makes it almost impossible to solve for an analytic expression or solution in τ, unless $\pi = 0$, in which case $\tau \to \infty$. Numerically though, it's quite trivial to isolate τ, especially in R.

Before we take that step, let's redefine $\psi_0 = \pi\texttt{ILA}(\texttt{x})/(w + \pi\texttt{ILA}(\texttt{x}))$ as the fraction of total (balance sheet) wealth that is initially pensionized. This ratio does not depend on τ or γ but is an implicit function of (r, m, b). Dividing equation (41) by π and then substituting $w = \texttt{ILA}(\texttt{x})\pi\frac{1-\psi_0}{\psi_0}$, leaves us with:

$$\left[\frac{\left(\texttt{ILA}(\texttt{x})\frac{1-\psi_0}{\psi_0} + \frac{1}{r}\right)e^{r\tau} - \frac{1}{r}}{\texttt{TLA}(\texttt{x} - b\ln\gamma, \tau)\, e^{r\tau}}\right]({}_\tau p_x)^{1/\gamma} \;-\; 1 \;=\; 0. \tag{42}$$

We have scaled the problem and eliminated a variable by focusing on ψ_0 instead of the duo (w, π), as it relates to computing the wealth depletion time. After much preparation, we are ready for our first computational step which is to locate the (numerical) value of τ that solves the above equation (42). This can be done in R using the built in `uniroot` function, because the left-hand side of equation (42) is monotonically decreasing in τ and only hits zero once, at least for realistic values of $x, r, m, b, \gamma, \psi_0$. In particular, we construct the function $f(t)$ via the following script:

```
f<-function(t){(((ILA(x,r,m,b)*(1-psi)/psi+1/r)*exp(r*t)-1/r)
*exp(exp((x-m)/b)*(1-exp(t/b)))^(1/gam))/
(TLA(x-b*log(gam),t,r,m,b)*exp(r*t))
-1}
```

This codes-up the left hand side of equation (42) and we are now ready to compute numerical results. We take an $x = 65$ year-old retiree, with a real pension entitlement of $\pi = \$25,000$ per year, under an $r = 2.5\%$ valuation rate, and Gompertz parameters $m = 81$ and $b = 11.5$. The full immediate life annuity factor `ILA(x,r,m,b)`, as well as the initial fraction of wealth that is pensionized ψ_0, and present value of the entire balance sheet, are computed by the following R-script.

```
> x<-65; r<-0.025; b<-11.5; m<-81;
> w<-100000; pi<-25000;
> psi<-ILA(x,r,m,b)*pi/(w+ILA(x,r,m,b)*pi)
> ILA(x,r,m,b); psi; w+ILA(x,r,m,b)*pi
[1] 12.21481
[1] 0.753312
[1] 405370.3
```

Although the initial liquid wealth is $w = \$100,000$, once the pre-existing pension income is added and valued at a factor multiple of ILA(65) $= 12.214$, the economic value of the consumer's balance sheet is $\$405,370$, of which $\psi_0 = 75.3\%$ is pensionized.

Finally, when $\gamma = 4$ for example, the command `uniroot` in R, with the lower bound of $t = 0$ (which is the earliest possible wealth depletion time) and upper bound of $t = \omega - x$, leads to:

```
> gam<-4; uniroot(f,lower=0,upper=omega-x)$root
[1] 20.82686
```

This is a wealth depletion time of 20.8 years. Conceptually, R starts searching at the last possible wealth depletion time, the biological end of the mortality table, and computes $f(t)$ at that extreme end-point. Given that $\pi > 0$ and $\psi_0 > 0$, $f(t) < 0$ at $t = \omega - x$. The algorithm then slowly reduces t until $f(t) = 0$. Thinking back to equation (40), t is the time at which the consumption rate is exactly equal to the pension income, which is a type of smooth pasting condition on consumption. We use the phrase *conceptually* instead of computationally because `uniroot` does something slightly more sophisticated, but the end result is the same. For calibration and testing purposes, here are some additional values of τ for lower levels of risk aversion γ, which results in earlier wealth depletion times.

```
> gam<-3; uniroot(f,lower=0,upper=omega-x)$root
[1] 18.92299
> gam<-2; uniroot(f,lower=0,upper=omega-x)$root
```

```
[1] 16.40704
> gam<-1; uniroot(f,lower=0,upper=omega-x)$root
[1] 12.63457
```

These numbers are in years. A 65 year-old retiree from the 1930 SSA cohort, with a coefficient of relative risk aversion (CRRA) of $\gamma = 1$, a.k.a. logarithmic utility, should rationally deplete wealth by the age of $x + \tau = 65 + 12.63 = 77.63$ and from then on, assuming they were still alive, live-off their Social Security benefits; for the earlier w, π and ψ values.

As far as interest (valuation) rates are concerned, its impact on the wealth depletion time is muted. Here is an example with (a lower) value of $r = 1\%$. The original parameters $x = 65$, $m = 81$ and $b = 11.5$ are retained for comparison, as is the value of $\gamma = 4$. The only argument or variable that must be modified in our code is r, which affects the $f(t)$ function directly via equation (42), as well as indirectly via ψ_0. As before, we report the annuity factor `ILA(65)`, as well as ψ_0.

```
> x<-65; r<-0.01; b<-11.5; m<-81; gam<-4;
> w<-100000; pi<-25000;
> psi<-ILA(x,r,m,b)*pi/(w+ILA(x,r,m,b)*pi)
> ILA(x,r,m,b); psi; w+ILA(x,r,m,b)*pi
[1] 14.01202
[1] 0.7779261
[1] 450300.5
> uniroot(f,lower=0,upper=omega-x)$root
[1] 20.19228
```

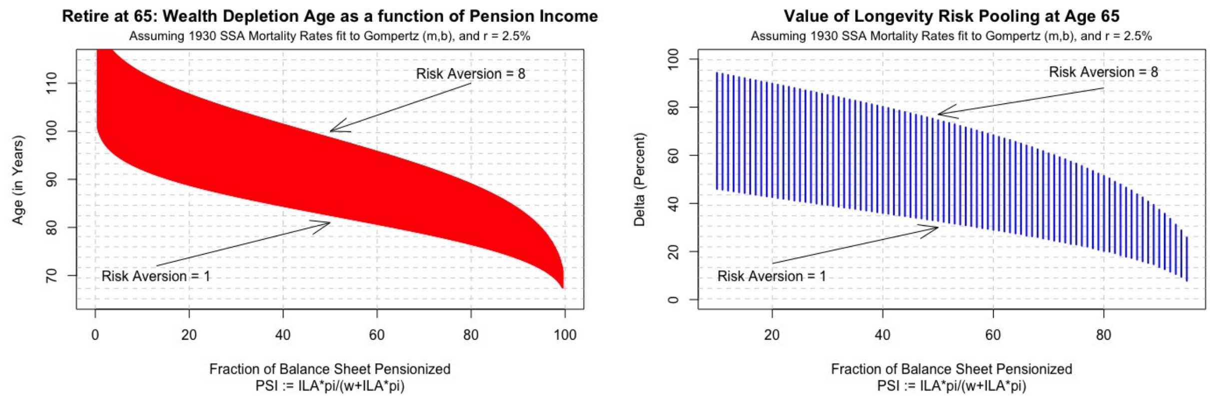
The present value of the entire balance is sheet is now $\$450,300$ (versus the $\$405,000$), and the fraction of wealth that is initially pensionized increases to $\psi_0 = 77.8\%$. The wealth depletion time only changes slightly, from 20.8 years to 20.19 years, which is a reduction of slightly more than six months. Stated differently, the age which you should "target" to run out of money – when you have a good pension – isn't very sensitive to interest rates; it's age 85 for the above pension income $\pi = \$25,000$ and risk aversion $\gamma = 4$, level.

Before we move on to solving for δ and the value of longevity pooling, we take the opportunity to *store* numerical values for the wealth depletion time over an entire range: $0 < \psi < 1$, so that they can be easily recalled and used later. In particular, we discretize ψ into units of 0.001 and compute the corresponding values of τ using the following R-script.

```
gam<-1; wdtg1<-c();
for (i in 1:995){psi<-i/1000
wdtg1[i]<-uniroot(f,lower=0,upper=omega-x)$root}
```

We stop the loop at 995 for reasons of numerical stability, since we know the value converges to zero when $\psi_0 = 1$. For example, irrespective of the actual values of (w, π), the wealth

Figure 2: Appendix Figure A1



depletion time for $\psi_0 = 0.2, \psi_0 = 0.5$ and $\psi_0 = 0.8$ are obtained from the 200th, 500th, and 800th element of the `wdtg1` vector. They are:

```
> wdtg1[200]; wdtg1[500]; wdtg1[800]
[1] 23.85256
[1] 17.59238
[1] 11.53844
```

We can do the same for a coefficient of relative risk aversion $\gamma = 8$, and denote the corresponding vector by `wdtg8` instead of the `wdtg1`

```
> gam<-8; wdtg8<-c();
> for (i in 1:995){psi<-i/1000
+ wdtg8[i]<-uniroot(f,lower=0,upper=65)$root}
> wdtg8[200]; wdtg8[500]; wdtg8[800]
[1] 42.64514
[1] 33.58799
[1] 23.99997
```

For higher levels of risk aversion, the wealth depletion time is rationally and cautiously later. We forced the R-script upper bound (in `uniroot`) to a higher value of $t = 65$ because the wealth depletion age at high levels of risk aversion and low levels of ψ_0 is close to ω.

To get a visual sense of wealth depletion ages, Figure #A1 (left) plots the two vectors over $\psi \in [0, 1]$. The upper curve is `wdtg8`, and the lower curve is `wdtg1`. We conclude this subsection by reminding readers (and users) that implicit in this figure (and our calculations) is the assumption that personal subjective discount rates ρ are unaffected by γ and are in fact set equal to the objective valuation rate $\rho = r$. Otherwise, equation (42) must be modified as per Charupat et. al (2012, pg. 300).

5.3 Equating Utilities

With the wealth depletion time readily available, we can now "think" of τ as being a function of the mortality parameters (x, m, b), the economic parameters (r, γ) and the pensionized fraction ψ. In fact, we will occasionally write and express τ as an explicit function $\tau(\psi)$, when we have to draw attention to it's dependence on ψ. Back to first principles, assuming an instantaneous CRRA utility function written as: $u(c) = (c^{1-\gamma} - 1)/(1-\gamma)$, complete annuitization of the original endowment w in the pair (w, π) results in discounted lifetime utility:

$$RHS = \; u\left(\frac{w}{(1+\kappa)\texttt{ILA(x)}} + \pi\right) = \frac{\texttt{ILA(x)}}{1-\gamma}\left(\frac{w}{(1+\kappa)\texttt{ILA(x)}} + \pi\right)^{1-\gamma} - \frac{\texttt{ILA(x)}}{1-\gamma}, \tag{43}$$

For completeness we include an *insurance loading* parameter $\kappa \geq 0$. Note that when converting wealth w into additional (retail) annuities, a commision or adverse selection cost of $1 - (1+\kappa)^{-1}$ of the amount annuitized might be charged, which for most of Milevsky & Huang (2018) was zero. In practice, additional annuity income will cost **more** than the (Social Security or group pension) annuity valuation factor `ILA(x)`, per \$1 unit of income. The *RHS* in equation (43) denotes the right hand side of a utility equalization process and many of the cumbersome coefficients and parameters will soon cancel.

Moving on to the left hand side of the utility equalization process in which an artificial $\hat{w} > w$ is invested at r and run-down or spent until the wealth depletion time τ, we have:

$$LHS \; = \int_0^{\tau} e^{-rt}\,({}_tp_x)\,u(c_t^{**})dt \; + \; u(\pi)\int_{\tau}^{\omega - x} e^{-rt}\,({}_tp_x)\,dt. \tag{44}$$

Maximal utility from $(\hat{w}, \pi)$ is obtained by consuming $c_t^{**} > \pi$ until time $t = \tau$ and then living off the pension π, once $t \geq \tau$.

We now recognize he second integral in equation (44) as the *delayed* life annuity factor, or the function $\texttt{DLA}(\texttt{x}, \tau)$ in R. The annuity equivalent wealth $\hat{w} > w$ doesn't appear explicitly in equation (44), but it's implicit in both τ, as well as $c_t^{**} \neq c_t^{*}$.

For ease of computation, we break-up equation (44) into LHS_1 and LHS_2, which after substituting the utility function $u(c)$, leads to:

$$LHS_1 \;\; = \;\; \int_0^{\tau} e^{-rt}\,({}_tp_x)\left[\frac{(c_t^{**})^{1-\gamma} - 1}{1-\gamma}\right]dt = \frac{\int_0^{\tau} e^{-rt}\,({}_tp_x)\,(c_t^{**})^{1-\gamma}dt}{1-\gamma} - \frac{\texttt{TLA}(\texttt{x},\tau)}{1-\gamma}, \tag{45}$$

$$LHS_2 \;\; = \;\; \frac{(\pi^{1-\gamma} - 1)\,\texttt{DLA}(\texttt{x},\tau)}{1-\gamma} = \frac{\pi^{1-\gamma}\,\texttt{DLA}(\texttt{x},\tau)}{1-\gamma} - \frac{\texttt{DLA}(\texttt{x},\tau)}{1-\gamma}. \tag{46}$$

And, since $\texttt{ILA} = \texttt{DLA} + \texttt{TLA}$, when the underlying parameters r, m, b are the same, the last items in equations (43), (45) and (46) cancel. Moreover, assuming $\gamma \neq 1$ (which we will return to), and after multiplying by $(1-\gamma)$ the fundamental utility equalization process leads to:

$$\int_0^{\tau} e^{-rt}\,({}_tp_x)\,(c_t^{**})^{1-\gamma}dt \; + \; \pi^{1-\gamma}\,\texttt{DLA}(\texttt{x},\tau) = \texttt{ILA(x)}\left(\frac{w}{(1+\kappa)\texttt{ILA(x)}} + \pi\right)^{1-\gamma}. \tag{47}$$

Once again we refer to these three items sequentially as LHS_1, LHS_2, RHS, and tackle each of them individually. First, we focus on LHS_1 and recall that the optimal consumption function $c_t^{**} = c_0^{**}({}_tp_x)^{1/\gamma}$, as per Charupat et al. (2012, pg. 301) and can be written as:

$$c_t^{**} \;=\; \left[\frac{(\hat{w}+\pi/r)e^{r\tau}-\pi/r}{\mathtt{TLA}(\mathtt{x}{-}b\ln\gamma,\tau)\,e^{r\tau}}\right]({}_tp_x)^{1/\gamma}, \tag{48}$$

where $\mathtt{TLA}$ in the denominator is an age-modified temporary life annuity factor with current age: $x - b\ln\gamma$, delayed for τ years. The other three variables are the usual (r, m, b), and not listed explicitly.

The item in square brackets of equation (48) is the optimal initial consumption rate c_0^{**} assuming an initial wealth of $\hat{w}$, not the original w. We pause here to emphasize (again) the difference between c_t^{**} versus c_t^{*}, which is critical. The former double-star is the optimal consumption strategy *after* we have replaced w with its annuity equivalent wealth $\hat{w}$. The latter single-star is the optimal consumption strategy given the initial wealth w. Note that both strategies also have different wealth depletion times, because they differ in the fraction of wealth that is pensionized: ψ. In particular, the relevant ratio is now, $\hat{\psi} = \mathtt{ILA}(\mathtt{x})\pi/(\hat{w} + \mathtt{ILA}(\mathtt{x})\pi) \leq \psi_0$ and the wealth depletion time: $\tau(\hat{\psi}) \geq \tau(\psi_0)$. See figure #1 (left panel) for an intuitive picture and reason.

Substituting c_t^{**} into what we called (the new) LHS_1 leads to:

$$LHS_1 \;=\; \left[\frac{(\hat{w}+\pi/r)e^{r\tau}-\pi/r}{\mathtt{TLA}(\mathtt{x}{-}b\ln\gamma,\tau)\,e^{r\tau}}\right]^{1-\gamma} \int_0^{\tau} e^{-rt}\,({}_tp_x)^{1/\gamma}dt, \tag{49}$$

which, after recognizing that the integral portion is (also) the definition of the modified *temporary* annuity factor, leads to a simpler:

$$LHS_1 \;=\; \left[\hat{w}e^{r\tau} + \frac{\pi}{r}(e^{r\tau}-1)\right]^{1-\gamma} \mathtt{TLA}^{\gamma}(\mathtt{x}{-}b\ln\gamma,\tau)\,e^{(\gamma-1)r\tau}. \tag{50}$$

Going back to our original definition of ψ_0, since $\hat{w} := (1+\delta)w$, and $\pi = w\frac{\psi_0}{(1-\psi_0)\mathtt{ILA}(\mathtt{x})}$, we can re-write the above LHS_1 as:

$$LHS_1 \;=\; w^{1-\gamma}\;\left[(1+\delta) + \frac{\psi_0(e^{r\tau}-1)}{re^{r\tau}(1-\psi_0)\mathtt{ILA}(\mathtt{x})}\right]^{1-\gamma} \mathtt{TLA}^{\gamma}(\mathtt{x}{-}b\ln\gamma,\tau). \tag{51}$$

We emphasize (yet again) the wealth depletion time $\tau(\hat{\psi})$ is for the "new" value of initial wealth $\hat{w}$, that is the "new" value of the pensionization fraction: $\psi_\delta = \psi_0/(\delta(1-\psi_0)+1)$. Reducing ψ will increase the value of τ. Along the same lines, replacing π with $w\frac{\psi_0}{(1-\psi_0)\mathtt{ILA}(\mathtt{x})}$ the LHS_2 can be written as:

$$LHS_2 \;= w^{1-\gamma}\left(\frac{\psi_0}{\mathtt{ILA}(\mathtt{x})(1-\psi_0)}\right)^{1-\gamma} \mathtt{DLA}(\mathtt{x},\tau) \tag{52}$$

where the $\mathtt{DLA}(\mathtt{x},\tau)$ is valued at the current age x, under the parameters r, m, b. After replacing π, the RHS in the utility equalization process is:

$$RHS = w^{1-\gamma}\,\mathtt{ILA}(\mathtt{x})\left(\frac{1}{(1+\kappa)\mathtt{ILA}(\mathtt{x})} + \frac{\psi_0}{(1-\psi_0)\mathtt{ILA}(\mathtt{x})}\right)^{1-\gamma}. \tag{53}$$

Dividing both sides of the utility equalization relationship, that is $LHS_1 + LHS_2$ as well as RHS, by $w^{1-\gamma}$ leads to the following relationship for δ:

$$[(1+\delta) \;+\; \boldsymbol{\alpha_1}]^{1-\gamma} \;=\; \boldsymbol{\alpha_2} - \boldsymbol{\alpha_3}. \tag{54}$$

where the three new α functions are defined as:

$$\boldsymbol{\alpha_1} \;=\; \frac{\psi_0(e^{r\tau}-1)}{re^{r\tau}(1-\psi_0)\texttt{ILA(x)}}, \tag{55}$$

$$\boldsymbol{\alpha_2} = \frac{\left(\frac{1}{\texttt{ILA(x)}(1+\kappa)} + \frac{\psi_0}{(1-\psi_0)\texttt{ILA(x)}}\right)^{1-\gamma}\texttt{ILA(x)}}{\texttt{TLA}^{\gamma}(\texttt{x}{-}b\ln\gamma,\tau)}, \quad \boldsymbol{\alpha_3} = \frac{\left(\frac{\psi_0}{\texttt{ILA(x)}(1-\psi_0)}\right)^{1-\gamma}\texttt{DLA}(\texttt{x},\tau)}{\texttt{TLA}^{\gamma}(\texttt{x}{-}b\ln\gamma,\tau)}. \tag{56}$$

All three α terms depend on the wealth depletion time τ, which itself depends on δ thru $\hat{\psi}$, so the solution process is obviously iterative. Taking the final step; since `ILA >= ILA` in equation (54) it's clear that $\alpha_2 \geq \alpha_3$, regardless of the value of κ in α_2. So, we take roots $1/(1-\gamma)$ of equation (54), and the value of longevity risk pooling collapses to:

$$\delta \;=\; [\boldsymbol{\alpha_2} - \boldsymbol{\alpha_3}]^{1/(1-\gamma)} \;-\; \boldsymbol{\alpha_1} \;-\; 1 \tag{57}$$

Compare this with the expression derived in Milevsky & Huang (2018). When $\pi = 0$, so that $\psi_0 = 0$, and the loading $\kappa = 0$, we have $\alpha_1 = \alpha_3 = 0$, and equation (57) collapses to:

$$\delta = \left[\frac{\texttt{ILA(x)}\left(\frac{1}{\texttt{ILA(x)}}\right)^{1-\gamma}}{\texttt{TLA}^{\gamma}(\texttt{x}{-}b\ln\gamma,\tau)}\right]^{1/(1-\gamma)} - 1 \;=\; \left(\frac{\texttt{ILA(x)}}{\texttt{ILA}(\texttt{x}-b\ln\gamma)}\right)^{\frac{\gamma}{1-\gamma}} - 1, \tag{58}$$

which recovers the correct value equation. Q.E.D.

In R, we use an iterative procedure for locating δ that satisfies equation (54), and not (57), mostly for reasons of numerical stability. Basically, we (1.) start a loop with a value of $\delta = i/1000$, (2.) compute the reduced ratio $\hat{\psi}_i$, and (3.) identify the new (higher) wealth depletion time $\tau(\hat{\psi}_i)$, and finally (4.) plug these values into the alphas and then equation (54) to check for equality. If both sides are equal to each other, or more precisely: $((1+\delta)+\alpha_1)^{1-\gamma} - \alpha_2 + \alpha_3 \leq 0$ (when $\gamma > 1$ and vice versa when $\gamma < 1$), we stop the search and report δ as the value of longevity risk pooling for a given ψ_0. If the condition is not met and utility is therefore not equalized for that δ, we increase the value of i (larger annuity equivalent wealth), create a new (lower) ψ_i and begin the process anew. Eventually the value of δ will be large enough so that utility is indeed equalized and we have converged to the value of longevity risk pooling in the presence of pre-existing annuity income.

```
x<-65; r<-0.025; b<-11.5; m<-81; gam<-5;
k<-0; w<-100000; pi<-8200;
psi0<-ILA(x,r,m,b)*pi/(w+ILA(x,r,m,b)*pi)
delta.0<-(ILA(x,r,m,b)/ILA(x-b*log(gam),r,m,b))^(gam/(1-gam))-1
S<-ceiling(delta.0*1000)
```

```
#The Algorithm will search from Delta = 0, upward.
for (i in 0:S){
  delta<-i/1000; psi<-psi0/(delta*(1-psi0)+1)
  f<-function(t){(((ILA(x,r,m,b)*(1-psi)/psi+1/r)*exp(r*t)-1/r)
  *exp(exp((x-m)/b)*(1-exp(t/b)))^(1/gam))/
  (exp(r*t)*TLA(x-b*log(gam),t,r,m,b))-1}
#Compute the Wealth Depletion Time (WDT)
  tau<-uniroot(f,lower=0,upper=omega-x)$root
#Compute the three Alpha Values
  alpha1<-(psi0*(exp(r*tau)-1))/(r*exp(r*tau)*(1-psi0)*ILA(x,r,m,b))
  alpha2<-(ILA(x,r,m,b)*(1/(ILA(x,r,m,b)*(1+k))+psi0/((1-psi0)
              *ILA(x,r,m,b)))^(1-gam))/TLA(x-b*log(gam),tau,r,m,b)^(gam)
  alpha3<-(DLA(x,tau,r,m,b)*(psi0/((1-psi0)*ILA(x,r,m,b)))^(1-gam))/
               TLA(x-b*log(gam),tau,r,m,b)^(gam)
#Test for Utility Equalization.
  if (((1+delta)+alpha1)^(1-gam)-alpha2+alpha3<=0){
    print("Analytic Delta Zero"); print(delta.0);
    print("Pensionized (Psi)"); print(psi0);
    print("Delta for given Psi:"); print(delta);
    break}}
```

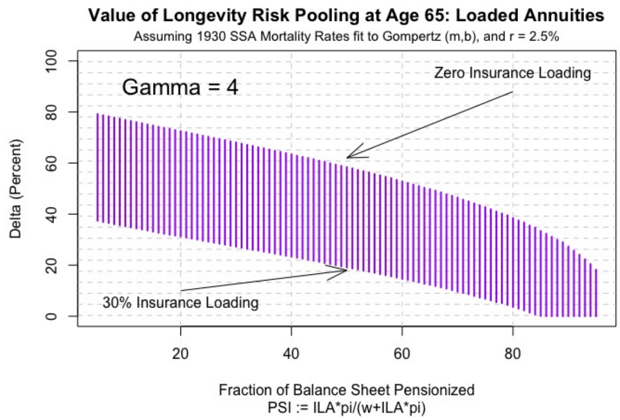
The algorithm initializes values of $x, r, m, b, \gamma, w, \pi$, as well as the loading factor k. The R-script computes ψ_0, which is the initial fraction pensionized, as well as the value of δ_0, which is the value of longevity risk pooling if $\pi = \psi = 0$. It's the upper bound on the value of $\hat{\psi}_\delta$ and becomes the maximal value for loop iteration. The algorithm gradually increases δ and by correspondingly reduces $\hat{\psi}_\delta = \psi_0/(\delta(1-\psi_0)+1)$ and computes the wealth depletion time $\tau(\hat{\psi}_\delta)$ for that value of δ. It then computes the three α values and tests for the utility equalization condition. If they are indeed equal, or more precisely equation (54) is less than or equal to zero (since we assume $\gamma > 1$), the algorithm stops and the most recent δ is our candidate.

We conclude with a few runs of the R-script. The parameters were selected to match Table 5.10 in Brown et al. (2001). Although they use "old" mortality, our choice is meant as a check on methodology. In practice, we would calibrate (m, b) to recent tables.

```
[1] "Analytic Delta Zero"
[1] 0.8730134
[1] "Pensionized (Psi)"
[1] 0.5004033
[1] "Delta for given Psi:"
[1] 0.634
```

We focus on $\gamma = 5$. The value of longevity risk pooling for an individual with no pre-existing annuity, $\psi_0 = 0$ using our notation, is $\delta_0 = 87.3\%$, compared with 85.5% in Brown et al.

Figure 3: Appendix Figure A2



More importantly, when the initial real pension income is set to $\pi = \$8,200$, which is a pensionization fraction of $\psi_0 = 0.5$, the value of pooling is $\delta_{0.5} = 63.4\%$, and compares quite favorably with their 62.3%, a case they called *half of initial wealth in pre-existing annuity.* Here are results of another run in R, under a value of $\gamma = 1.01$, to get close to logarithmic utility.

```
[1] "Analytic Delta Zero"
[1] 0.5010502
[1] "Pensionized (Psi)"
[1] 0.5004033
[1] "Delta for given Psi:"
[1] 0.326
```

The values in Brown et al. are 50.2% for no preexisting annuity income and 33.0% for $\psi_0 = 0.5$. We obtain similar results for $\gamma = 2$ and $\gamma = 10$; the two other values they reported. On a separate note, using our algorithm with $m = -b \ln \lambda b$ and $b \to \infty$, which implies Gompertz converges to exponential, we comfortably reproduce the numbers in table #1 of Milevsky & Huang (2018). Finally, when $\kappa > 0\%$ insurance loading, the value of pooling δ is much lower and is illustrated in figure #A2 with $\gamma = 4$ and $\kappa = 30\%$.